Selected Ph.D. Microeconomics 1 Proofs

Brian Murphy[*][†]

Properties

- **Utility:** Strictly increasing, strictly quasiconcave, and continuous.
- **Indirect Utility:** Continuous, homogeneous of degree (HOD) zero in (p, y), increasing in y, decreasing in p, quasiconvex in (p, y), and satisfies Roy's identity.
- **Expenditure Function:** Continuous, zero when $u = 0$, strictly increasing in p, HOD 1 in p, concave in p, and satisfies Shephard's Lemma.
- **Production Function:** Strictly increasing, strictly quasiconcave, and (typically) exhibits constant returns to scale (CTS).
- **Cost Function:** Zero when $y = 0$, continuous, increasing in w, HOD 1 in w, concave in w, and satisfies Shephard's Lemma.
- **Conditional Input Demands:** Denoted $x(w, y)$; HOD 1 in w and with a negative semidefinite substitution matrix.
- **Profit Function:** Increasing in output prices p, decreasing in input prices w, HOD 1 in (p, w), convex in (p, w), and (if f is strictly concave) satisfies Hotelling's Lemma.
- **Output Supply & Input Demand:** Homogeneous of degree zero:

  $$y(tp, tw) = y(p, w), \quad x(tp, tw) = x(p, w),$$

  with own price effects

  $$\frac{\partial y(p, w)}{\partial p} \geq 0, \quad \frac{\partial x(p, w)}{\partial p} \leq 0.$$

- **Excess Demand Functions:** Continuous, HOD zero in p, and satisfy Walras' law.

Profit Function Convexity

Let

$$p'' = tp + (1 - t)p', \quad t \in [0, 1].$$

We want to show that

$$\pi(p'') \leq t\,\pi(p) + (1 - t)\,\pi(p').$$

Assume that y is the profit-maximizing production plan at price p, and y' is the corresponding plan at p'. Let y'' be the production plan for p''. Then

$$\begin{aligned}\pi(p'') &= \left[tp + (1 - t)p'\right] \cdot y'' \\ &= t\,p \cdot y'' + (1 - t)\,p' \cdot y''.\end{aligned}$$

[*]These notes are from my time as a student in the University of Houston PhD Economics program.
[†]Typos may exist in these notes. If any are found, please contact me.

Since y and y' are optimal at p and p' respectively, we have

$$t\,p \cdot y \leq t\,p \cdot y'' \quad \text{and} \quad (1-t)\,p' \cdot y' \leq (1-t)\,p' \cdot y''.$$

Adding these inequalities gives the desired convexity:

$$\pi(p'') \leq t\,\pi(p) + (1-t)\,\pi(p').$$

□

Expenditure Function Concavity

Let $e(p,u)$ be the expenditure function. For

$$p'' = tp + (1-t)p',$$

we want to show

$$e(p'',u) \geq t\,e(p,u) + (1-t)e(p',u).$$

Since the expenditure function is defined by

$$e(p,u) = p \cdot x^H(p,u),$$

where $x^H(p,u)$ is the Hicksian demand, note that

$$e(p'',u) = p'' \cdot x^H(p'',u) = t\,p \cdot x^H(p'',u) + (1-t)\,p' \cdot x^H(p'',u).$$

Because $x^H(p,u)$ minimizes expenditure, it follows that

$$t\,p \cdot x^H(p'',u) \geq t\,e(p,u) \quad \text{and} \quad (1-t)\,p' \cdot x^H(p'',u) \geq (1-t)\,e(p',u).$$

Thus,

$$e(p'',u) \geq t\,e(p,u) + (1-t)e(p',u).$$

□

Why is the Cost Function Concave?

Since the cost function is equivalent to the expenditure function, the same concavity proof applies. In other words, for

$$p'' = tp + (1-t)p'$$

and any utility level $\bar{u}$,

$$e(p'',\bar{u}) \geq t\,e(p,\bar{u}) + (1-t)e(p',\bar{u}),$$

which implies the cost function is concave in p. □

Profit Function Properties

1. **Nondecreasing in Output Prices:**
$$\frac{\partial \pi}{\partial p} \geq 0.$$
2. **Nonincreasing in Input Prices:**
$$\frac{\partial \pi}{\partial w} \leq 0.$$
3. **Homogeneity of Degree 1:**
$$\pi(t\,w, t\,p) = t\,\pi(w,p).$$
4. **Convexity:** As shown above.
5. **Hotelling's Lemma:** If the production function $f(\cdot)$ is strictly concave, then Hotelling's Lemma holds.

Marshallian Demands and Elasticities

Marshallian demand elasticities are given by:

$$\text{Own price elasticity: } \frac{\partial x_i}{\partial p_i} \cdot \frac{p_i}{x_i} \quad \text{(usually negative)},$$

$$\text{Cross price elasticity: } \frac{\partial x_i}{\partial p_j} \cdot \frac{p_j}{x_i} \quad \text{(positive for substitutes, negative for complements)},$$

$$\text{Income elasticity: } \frac{\partial x_i}{\partial y} \cdot \frac{y}{x_i}.$$

Additional Assumptions/Properties

- **Utility:** Continuous, strictly increasing, and strictly quasiconcave.
- **Indirect Utility:** Continuous, HOD zero in (p, y), increasing in y, decreasing in p, quasiconvex in (p, y), satisfies Roy's identity.
- **Expenditure Function:** Continuous, zero when $u = 0$, strictly increasing in p, HOD 1 in p, and concave in p.
- **Production Function:** Continuous, strictly increasing, and strictly quasiconcave.
- **Cost Function:** Zero when $y = 0$, continuous, increasing in w, HOD 1 in w, concave in w, satisfies Shephard's Lemma.
- **Conditional Input Demands:** $x(w, y)$ is HOD 1 in w with a negative semidefinite substitution matrix.
- **Profit Function:** Increasing in p, decreasing in w, HOD 1 in (p, w), convex in (p, w), and (if f is strictly concave) satisfies Hotelling's Lemma.
- **Output Supply & Input Demand:**
  1. Homogeneous of degree zero:
  $$y(tp, tw) = y(p, w), \quad x(tp, tw) = x(p, w).$$
  2. Own price effects:
  $$\frac{\partial y(p, w)}{\partial p} \geq 0, \quad \frac{\partial x(p, w)}{\partial w} \leq 0.$$
- **Excess Demand Functions:** Continuous, HOD zero in p, and satisfy Walras' law.

First Welfare Theorem Proofs

FWT with Production

Claim: Every Pareto Efficient Allocation (PEA) is a Walrasian Equilibrium Allocation (WEA).

Proof. Suppose (x, y) is a WEA at p^* but not Pareto efficient. Then there exists a feasible allocation $(\hat{x}, \hat{y})$ such that

$$u(\hat{x}) \geq u(x)$$

for all consumers. This implies

$$p^* \cdot \hat{x} \geq p^* \cdot x.$$

Summing over consumers,

$$p^* \cdot \sum \hat{y}^i \geq p^* \cdot \sum y^i,$$

which contradicts the assumption that firms maximize profit. □

FWT without Production

Suppose an allocation x is not Pareto efficient. Then there exists $\hat{x}$ such that

$$u(\hat{x}) \geq u(x) \quad \text{and} \quad p \cdot \hat{x} \geq p \cdot x.$$

Since all agents face binding constraints, this leads to a contradiction in consumer optimality.

Second Welfare Theorem without Production

If x is Pareto efficient and feasible (i.e. $\sum x_i = \sum e_i$), then by monotonicity and feasibility, there exists an allocation that is Pareto efficient. (The full details are omitted here for brevity.)

Existence of Utility

Let e be the vector of ones and suppose $u(x)$ is defined on $\mathbb{X}$. Define

$$A = \{t \geq 0 \mid te \in \mathbb{X}\},$$
$$B = \{t \geq 0 \mid te \notin \mathbb{X}\}.$$

If there exists $t^* \in A \cap B$, then we define $u(x) = t^*$. Continuity of preferences implies that both A and B are closed. Monotonicity shows that if $t \in A$, then all $t' > t$ are in A. Hence, we may write $A = [t_*, \infty)$ and $B = [0, t_*]$. Completeness guarantees $A \cup B = [0, \infty)$, and uniqueness follows by monotonicity. $\square$

Slutsky Equation

The Slutsky equation can be written as:

$$\frac{\partial x_i(p,y)}{\partial p_j} = \frac{\partial x_i^H(p,u^*)}{\partial p_j} - x_i(p,y) \cdot \frac{\partial x_i(p,y)}{\partial y}.$$

Since the Hicksian demand satisfies

$$x_i^H(p,u) = x_i^H\big(p, e(p,u)\big),$$

differentiating with respect to p_j yields

$$\frac{\partial x_i^H}{\partial p_j} = \frac{x_i(p, e(p,u))}{p_j} + \frac{\partial x_i(p, e(p,u))}{\partial y} \cdot \frac{\partial e(p,u)}{\partial p_j}.$$

Noting that

$$\frac{\partial e(p,u)}{\partial p_j} = x_j^H(p,u),$$

we obtain the stated form.

Hotelling's Lemma

Consider the profit maximization problem

$$\max\ \pi(q, x_1, x_2, \dots; p, w_1, w_2, \dots) = pq - w_1 x_1 - w_2 x_2 \quad \text{subject to } f(x_1,x_2) \geq q.$$

Let the constraint be written as

$$G(x_1, x_2, q) = f(x_1, x_2) - q = 0.$$

Denote the profit function by $V(p, w_1, w_2)$. Then Hotelling's Lemma gives

$$\frac{\partial V}{\partial p} = q \quad \text{(output supply)} \quad \text{and} \quad \frac{\partial V}{\partial w_i} = -x_i(p,w) \quad \text{(input demand)}.$$

Shephard's Lemma for Consumers

For the consumer problem, consider the expenditure minimization:

$$\min_x \ e(x,p) = p_1x_1 + p_2x_2 \quad \text{subject to} \quad u(x_1,x_2) = \bar{u}.$$

Define

$$G(x_1,x_2,u) = u(x_1,x_2) - \bar{u}.$$

Then, by Shephard's Lemma,

$$\frac{\partial e(x,p)}{\partial p_i} = x_i^H(p,u).$$

Shephard's Lemma for Producers

For the producer problem, consider the cost minimization:

$$\min_x \ c(y,w) = w_1x_1 + w_2x_2 \quad \text{subject to} \quad f(x_1,x_2) \geq q.$$

Define

$$G(x_1,x_2,q) = f(x_1,x_2) - q = 0.$$

Then, by Shephard's Lemma,

$$\frac{\partial c(y,w)}{\partial w_i} = x_i(q,w).$$

Welfare Theorems

FWT $\Rightarrow$ WEA is PE

("Pareto Efficiency implies Walrasian Equilibrium")

If $\overline{x}$ is Pareto efficient (PE) then it is also a Walrasian equilibrium allocation (WEA) because feasibility ($\sum \overline{x}_i = \sum e_i$) combined with optimality prevents any deviation.

FWT with Production

Suppose (x,y) is a WEA at p^* but not Pareto efficient. Then

$$\sum x^i = \sum y^i + \sum e^i.$$

Since it is not PE, there exists a feasible allocation $(\hat{x},\hat{y})$ such that

$$u^i(\hat{x}^i) \geq u^i(x^i).$$

This implies

$$p^* \cdot \hat{x}^i \geq p^* \cdot x^i,$$

and consequently

$$p^* \cdot \sum \hat{y}^i > p^* \cdot \sum y^i,$$

which contradicts profit maximization.

SWT with Production

Consider an economy with modified endowments $\overline{e} = (u^i, \hat{x}^i, e^i, \hat{Y}^i)$ where each consumer's endowment is augmented by the production set $\hat{Y}^i$. Since firms earn nonnegative profits, each consumer can afford his/her endowment vector. Thus,

$$u^i(\overline{x}^i) \geq u^i(\hat{x}^i).$$

For some aggregate production vector $\hat{y}$, the allocation $(\overline{x}, \hat{y})$ is feasible in the original economy:

$$\begin{aligned}
\sum \overline{x}^i &= \sum \hat{x}^i + \sum y^j \\
&= \sum \hat{x}^i + \sum (y^j - \hat{y}^j) \\
&= \sum \hat{x}^i - \sum \hat{y}^j + \sum \hat{y}^j \\
&= \sum e^i + \sum \hat{y}^j.
\end{aligned}$$

Strict quasiconcavity forces $\hat{y}^i = \hat{y}^*$ (otherwise averaging would improve utility), which in turn implies zero profit.

Consumer Choice Axioms

1. **Completeness:** For any two bundles x_1 and x_2, either $x_1 \succeq x_2$, $x_2 \succeq x_1$, or $x_1 \sim x_2$.
2. **Transitivity:** If $x_1 \succeq x_2$ and $x_2 \succeq x_3$, then $x_1 \succeq x_3$.
3. **Continuity:** Preferences are continuous; small changes do not lead to abrupt reversals.
4. **Strict Monotonicity:** If $x_1 \geq x_2$ (with at least one strict inequality), then $x_1 \succ x_2$.
5. **Strict Convexity:** For any distinct bundles x_1 and x_0 with $x_1 \succeq x_0$ and for all $t \in (0,1)$,

$$t\,x_1 + (1-t)x_0 \succ x_0.$$

Utility Function and Existence

A function $u : \mathbb{R}^n_+ \to \mathbb{R}$ represents preferences if

$$u(x') \geq u(x) \iff x' \succeq x.$$

Under the assumptions of completeness, transitivity, and continuity, such a utility representation exists.

Existence Proof

Let e be the vector of ones and define

$$A = \{t \geq 0 \mid te \in \mathbb{X}\},$$
$$B = \{t \geq 0 \mid te \notin \mathbb{X}\}.$$

If there exists $t^* \in A \cap B$, then we define $u(x) = t^*$. Continuity of preferences implies both A and B are closed. Monotonicity ensures that if $t \in A$, then every $t' > t$ is also in A. Hence, one can write $A = [t_*, \infty)$ and $B = [0, t_*]$. Completeness guarantees $A \cup B = [0, \infty)$, and by monotonicity the intersection is a singleton. $\square$

Indirect Utility

The indirect utility function is continuous, HOD zero in (p, y), strictly increasing in y, decreasing in p, quasiconvex in (p, y), and satisfies Roy's identity.

Proof of Homogeneity of Degree Zero

For any scalar $t > 0$,

$$v(tp, ty) = v(p, y),$$

since scaling both prices and income does not change the feasible set. $\square$