Notes on Macroeconomics II

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*These notes are from my time as a student in the University of Houston PhD Economics program.
†Typos may exist in these notes. If any are found, please contact me.

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2.1 Introduction

In modeling real-life individuals, there are two distinct approaches: one in which individuals live a finite number of periods, and another in which they live forever. In addition, there exist two alternative methods for solving dynamic optimization problems. The first is the *sequential method*, which involves maximizing over sequences, and the second is the *recursive method* (or *dynamic programming method*), which involves solving functional equations.

2.2 Sequential Methods

2.2.1 Finite Horizon: Consumer Choice

Consider a consumer who must decide on a consumption stream for T periods. The consumer's preference ordering over consumption streams can be represented by a utility function, denoted by

$$U(c_0, c_1, c_2, \ldots, c_T),$$

and is normally assumed to be additively separable:

$$U(c_0, c_1, c_2, \ldots, c_T) = \sum_{t=0}^{T} \beta^t u(c_t),$$

where β^t are stationary discounting weights and it is assumed that $0 < \beta < 1$.

2.2.2 Finite Horizon: The Neoclassical Growth Model

Let us state the dynamic optimization problem for the neoclassical growth model over a finite horizon. The planner's problem is:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t)$$

subject to the constraints:

$$c_t + k_{t+1} \leq f(k_t) \equiv F(k_t, N) + (1-\delta)k_t, \quad \forall t = 0, \ldots, T, \tag{1}$$

$$c_t \geq 0, \quad \forall t = 0, \ldots, T, \tag{2}$$

$$k_{t+1} \geq 0, \quad \forall t = 0, \ldots, T, \tag{3}$$

$$k_0 > 0 \quad \text{given.} \tag{4}$$

This is a planning problem where savings yield production through the function $f(k)$.

2.3 Solution Methodology

We assume that the utility function u is strictly increasing, so that no goods are wasted and the resource constraint binds. In this context, the Kuhn-Tucker theorem is used to characterize the solution. It is also useful to assume that

$$\lim_{c \to 0} u'(c) = \infty,$$

which implies that $c_t = 0$ for any t cannot be optimal. Hence, the non-negativity constraint on consumption can be disregarded in the analysis.

Two formulations of the Lagrangian are considered:

2.3.1 Formulation A

$$L = \sum_{t=0}^{T} \beta^t \left[u(c_t) - \lambda_t \left(c_t + k_{t+1} - f(k_t) \right) + \mu_t k_{t+1} \right].$$

2.3.2 Formulation B

$$L = \sum_{t=0}^{T} \beta^t \left[u(f(k_t) - k_{t+1}) + \mu_t k_{t+1} \right],$$

given our assumptions on u.

2.4 First Order Conditions and Kuhn-Tucker Conditions

The first order conditions (FOCs) for this problem are:

$$\frac{\partial L}{\partial c_t} = \beta^t \left[u'(c_t) - \lambda_t \right] = 0, \quad t = 0, \ldots, T, \tag{5}$$

$$\frac{\partial L}{\partial k_{t+1}} = -\beta^t \lambda_t + \beta^t \mu_t + \beta^{t+1} \lambda_{t+1} f'(k_{t+1}) = 0, \quad t = 0, \ldots, T-1. \tag{6}$$

For period T, we have:

$$\frac{\partial L}{\partial k_{T+1}} = -\beta^T \lambda_T + \beta^T \mu_T = 0.$$

The Kuhn-Tucker conditions that accompany these FOCs are:

$$\mu_t k_{t+1} = 0, \quad t = 0, \ldots, T, \tag{7}$$

$$\lambda_t \geq 0, \quad t = 0, \ldots, T, \tag{8}$$

$$k_{t+1} \geq 0, \quad t = 0, \ldots, T, \tag{9}$$

$$\mu_t \geq 0, \quad t = 0, \ldots, T. \tag{10}$$

Since $\lambda_t = u'(c_t) > 0$ for all t, and from the terminal condition $\lambda_T = \mu_T$, it follows from the complementary slackness condition that $k_{T+1} = 0$. Consequently, the Euler Equation (EE) is derived as:

$$u'(c_t) = \beta u'(c_{t+1}) f'(k_{t+1}).$$

This Euler Equation is central to the dynamic optimization problem, where we need to solve for $2T$ unknowns (namely, $\{k_1, \ldots, k_T\}$ and $\{c_0, \ldots, c_T\}$) subject to $T - 1$ Euler Equations and the terminal condition $k_{T+1} = 0$.

2.5 Interpretation of the Euler Equation

The Euler Equation,

$$u'(c_t) = \beta u'(c_{t+1}) f'(k_{t+1}),$$

can be interpreted as follows:

- $u'(c_t)$ represents the marginal cost of saving, as it is the utility lost when investing one more unit.
- $f'(k_{t+1})$ denotes the return on the invested unit, measured by the increase in next period's consumption.
- $\beta u'(c_{t+1})$ captures the discounted marginal benefit in terms of utility for an increase in consumption in the next period.

Because the utility function u is concave, equating the marginal cost of saving to the marginal benefit is a necessary condition for an optimal decision.

2.6 A General Example: Logarithmic Utility and Cobb-Douglas Technology

2.6.1 Model Setup

Consider the case where the utility function and the production function take the following forms:

$$u(c) = \log(c), \quad f(k) = k^\alpha,$$

with the additional assumption of full depreciation, i.e., $\delta = 1$. Suppose the planning horizon is $T = 2$ and that k_0 is given.

2.6.2 Solution for the Two-Period Model

We aim to solve for c_0, c_1, k_1, and k_2. First, note that by the terminal condition we have:

$$k_2 = 0.$$

The Euler Equation for $t = 0$ is:

$$u'(c_0) = \beta u'(c_1) f'(k_1).$$

Substituting the derivatives, we obtain:

$$\frac{1}{c_0} = \beta \alpha k_1^{\alpha-1} \frac{1}{c_1},$$

which implies that

$$c_1 = \alpha\beta k_1^{\alpha-1} c_0.$$

The budget constraint for period $t = 1$ is given by:

$$c_1 = k_1^\alpha - k_2.$$

Since $k_2 = 0$, this simplifies to:

$$c_1 = k_1^\alpha.$$

Combining these results leads to:

$$k_1 = \alpha\beta c_0.$$

Moreover, the budget constraint for period $t = 0$ reads:

$$c_0 = k_0^\alpha - k_1.$$

Substituting for k_1, we have:

$$c_0 = k_0^\alpha - \alpha\beta c_0.$$

Rearranging yields:

$$c_0 (1 + \alpha\beta) = k_0^\alpha,$$

and thus,

$$c_0 = \frac{1}{1 + \alpha\beta} k_0^\alpha.$$

Consequently, using the relation $k_1 = \alpha\beta c_0$, we deduce:

$$k_1 = \frac{\alpha\beta}{1 + \alpha\beta} k_0^\alpha.$$

This shows that a proportion $\frac{\alpha\beta}{1+\alpha\beta}$ of output is saved, while the remainder is consumed.

2.7 Sequential Methods

2.7.1 Finite Horizon

Now consider the T periods problem. The Euler Equation is now

$$\frac{1}{c_t} = \beta\alpha k_{t+1}^{\alpha-1}\frac{1}{c_{t+1}}. \tag{11}$$

Using the assumed functional forms, we have

$$\frac{1}{k_t^\alpha - k_{t+1}} = \frac{\beta\alpha k_{t+1}^{\alpha-1}}{k_{t+1}^\alpha - k_{t+2}}, \quad \forall t < T, \tag{12}$$

and

$$k_{T+1} = 0. \tag{13}$$

2.8 Solutions

2.8.1 Method 1

Redefine variables by letting

$$z_t = \frac{k_{t+1}}{k_t^\alpha}. \tag{14}$$

Rearrange terms in the Euler Equation:

$$k_{t+1}^\alpha - k_{t+2} = \beta\alpha k_{t+1}^{\alpha-1}\left[k_t^\alpha - k_{t+1}\right]. \tag{15}$$

Dividing by k_{t+1}^α yields

$$\frac{k_{t+1}^\alpha}{k_{t+1}^\alpha} - \frac{k_{t+2}}{k_{t+1}^\alpha} = \beta\alpha\frac{k_{t+1}^{\alpha-1}}{k_{t+1}^\alpha}\left[k_t^\alpha - k_{t+1}\right]. \tag{16}$$

That is,

$$1 - z_{t+1} = \beta\alpha k_{t+1}^{-1}\left[k_t^\alpha - k_{t+1}\right]. \tag{17}$$

Recognizing that

$$k_{t+1}^{-1}\left[k_t^\alpha - k_{t+1}\right] = \frac{1}{z_t} - 1, \tag{18}$$

we obtain

$$1 - z_{t+1} = \beta\alpha\left(\frac{1}{z_t} - 1\right). \tag{19}$$

Thus, the recursive relationship for z_t becomes

$$z_t = \frac{\beta\alpha}{1 + \beta\alpha - z_{t+1}}. \tag{20}$$

Going backward from the terminal condition,

$$z_T = \frac{k_{T+1}}{k_T^\alpha} = 0, \tag{21}$$

we have

$$z_{T-1} = \frac{\beta\alpha}{1+\beta\alpha}, \tag{22}$$

and

$$z_{T-2} = \frac{\beta\alpha}{1 + \beta\alpha - \frac{\beta\alpha}{1+\beta\alpha}}. \tag{23}$$

More generally, one can write

$$z_t = \beta\alpha \frac{1-(\beta\alpha)^{T-t}}{1-(\beta\alpha)^{T-t+1}}. \tag{24}$$

Given the definition of z_t, we obtain

$$k_{t+1} = \beta\alpha \frac{1-(\beta\alpha)^{T-t}}{1-(\beta\alpha)^{T-t+1}}\, k_t^{\alpha}. \tag{25}$$

This result shows that the fraction of output saved is time variant, reflecting a marginal propensity to save that depends on the distance between period t and the terminal period T.

2.9 General Case: No Closed Form Solution

If we now assume that $\delta < 1$, then a closed form solution is no longer available, and one must resort to numerical methods. In this case, the Euler Equation becomes:

$$\frac{1}{k_t^{\alpha} - k_{t+1}} = \frac{\beta\left[\alpha k_{t+1}^{\alpha-1} + (1-\delta)\right]}{k_{t+1}^{\alpha} - k_{t+2}}, \quad \forall t < T. \tag{26}$$

A common approach to solve this problem is the "shooting algorithm":

1. Guess an initial value for k_1.
2. Use the Euler Equation to generate a candidate sequence.
3. Check the terminal condition on k_{T+1}. If $k_{T+1} \simeq 0$, the solution is acceptable; if not, update the guess and repeat the process.

A natural initial guess may be obtained from the solution corresponding to the case $\delta = 1$, and the update can be performed using methods such as the bisection method or Newton's method.

2.10 Sequential Methods: Infinite Horizon

In many applications, it is of interest to consider the limit as $T \to \infty$, which is motivated both by practical concerns and by considerations such as dynastic planning. However, infinite horizon problems present additional issues:

- A solution may not exist.
- The problem may be ill defined.

2.10.1 Issues in the Infinite Horizon Setting

One prominent issue is that of unbounded utility. For the continuity of the objective function, it is necessary that the utility be bounded; otherwise, one could encounter different consumption sequences that yield infinite utility, making it impossible to rank preferences. This necessitates imposing additional requirements on both the specification of preferences and the technology.

2.11 Example 1: Utility

Consider a constant consumption plan $\{c_t\}_{t=0}^{\infty} = \{\bar{c}\}_{t=0}^{\infty}$. The total utility of this stream is

$$U = \sum_{t=0}^{\infty} \beta^t u(\bar{c}).$$

For this series to converge and hence for U to be finite, it is necessary that $\beta < 1$.

2.12 Example 2: Utility and Technology

Suppose that the production function and utility function are given by

$$f(k_t) = Rk_t \quad \text{and} \quad u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}.$$

Since $c_t \leq k_0 R^t$, substituting into the utility function yields

$$\begin{aligned} U &= \sum_{t=0}^{\infty} \beta^t \frac{(k_0 R^t)^{1-\sigma} - 1}{1-\sigma} \\ &= \sum_{t=0}^{\infty} \frac{(\beta R^{1-\sigma})^t k_0^{1-\sigma} - \beta^t}{1-\sigma} \\ &= \frac{k_0^{1-\sigma}}{1-\sigma} \sum_{t=0}^{\infty} (\beta R^{1-\sigma})^t - \frac{1}{1-\sigma} \sum_{t=0}^{\infty} \beta^t. \end{aligned} \tag{27}$$

Evaluating these geometric series gives

$$U = \frac{k_0^{1-\sigma}}{1-\sigma} \frac{1}{1-\beta R^{1-\sigma}} - \frac{1}{(1-\beta)(1-\sigma)}. \tag{28}$$

Thus, for U to be finite, we require that

$$\beta R^{1-\sigma} < 1.$$

2.13 Infinite Horizon and the Transversality Condition

In the finite horizon case, the terminal condition $k_{T+1} = 0$ provides a necessary boundary condition for solving the system of difference equations. For the infinite horizon problem, the economy continues indefinitely and the Euler equations alone do not uniquely determine a solution. The missing condition is analogous to $k_{T+1} = 0$ and is called the *transversality condition* (TVC).

2.13.1 Intuition and Sufficiency

The TVC expresses the idea that it is not optimal for the consumer to choose a capital sequence such that the present-value shadow value of capital remains positive as time tends to infinity. If it did, this would imply excessive saving, and a reduction in saving would lead to higher utility. Although we do not prove the necessity of the TVC here, we present a sufficiency condition.

Sufficiency Theorem. Consider the programming problem

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(k_t, k_{t+1}) \tag{29}$$

subject to $k_{t+1} \geq 0$ for all t. If a sequence $\{k_{t+1}^*\}_{t=0}^{\infty}$ and multipliers $\{\mu_{t+1}^*\}_{t=0}^{\infty}$ satisfy:

1. $k_{t+1}^* \geq 0$ for all t,
2. the Euler Equation:

$$F_2(k_t^*, k_{t+1}^*) + \beta F_1(k_{t+1}^*, k_{t+2}^*) + \mu_t^* = 0, \quad \forall t,$$

3. $\mu_t^* \geq 0$ and $\mu_t^* k_{t+1}^* = 0$ for all t, and

4. the transversality condition:

$$\lim_{t\to\infty} \beta^t F_1(k_t^*, k_{t+1}^*) k_t^* = 0,$$

and if $F(x, y)$ is concave in (x, y) and increasing in its first argument, then $\{k_{t+1}^*\}_{t=0}^{\infty}$ maximizes the objective function.

2.13.2 Example (TVC)

Consider the example with log utility, Cobb-Douglas production, and full depreciation, where

$$F(k_t, k_{t+1}) = \log(c_t) = \log(k_t^\alpha - k_{t+1}). \tag{30}$$

The transversality condition now reads

$$\lim_{t\to\infty} \beta^t \frac{\alpha k_t^{\alpha-1} k_t}{k_t^\alpha - k_{t+1}}. \tag{31}$$

This can be expressed as

$$\lim_{t\to\infty} \beta^t \alpha \frac{k_t^{\alpha-1} k_t}{k_t^\alpha} \frac{k_t^\alpha}{k_t^\alpha - k_{t+1}}. \tag{32}$$

Recalling that

$$z_t = \beta\alpha \frac{1 - (\beta\alpha)^{T-t}}{1 - (\beta\alpha)^{T-t+1}}, \tag{33}$$

it follows that

$$\lim_{t\to\infty} z_t = \beta\alpha. \tag{34}$$

Therefore,

$$\lim_{t\to\infty} \beta^t \frac{\alpha}{1 - \alpha\beta} = 0, \tag{35}$$

so that the TVC holds and all conditions of the theorem are satisfied.

3 Dynamic Programming

3.1 Why Dynamic Programming?

Dynamic programming offers significant advantages in both conceptual understanding and computational implementation. The search for an entire decision sequence may be impractical. Instead, dynamic programming allows decisions to be made recursively—time period by time period—making the analysis of dynamic problems more tractable. This approach is particularly powerful in stationary problems, where the structure of the decision problem remains unchanged over time.

3.2 The Infinite-Horizon Problem

Consider an agent at time $t = 0$ choosing an infinite consumption stream given an initial capital stock k_0. Suppose that at a later time, say $t = T$, the agent is faced with the same problem with a new initial capital stock k_T. If the agent's plan remains optimal upon reoptimization, this indicates the stationarity of the problem—the only relevant state variable is the capital stock.

3.3 The Bellman Equation

We define the value function $v(k)$ as the maximum attainable utility starting from capital stock k. At time $t = 0$, the problem is:

$$v(k_0) = \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{36}$$

subject to

$$c_t + k_{t+1} = F(k_t) + (1-\delta)k_t, \tag{37}$$

$$k_0 > 0 \quad \text{given.} \tag{38}$$

Here, $v(k_0)$ represents the value (in terms of utility) of having an initial capital stock k_0, assuming optimal decisions from $t = 1$ onward. Similarly, for any $k_1 > 0$,

$$v(k_1) = \max_{\{c_t, k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^t u(c_t) \tag{39}$$

subject to

$$c_t + k_{t+1} = F(k_t) + (1-\delta)k_t. \tag{40}$$

Thus, we can write the Bellman equation at time $t = 0$ as

$$v(k_0) = \max_{\{c_0, k_1\}} \{u(c_0) + \beta v(k_1)\} \tag{41}$$

subject to

$$c_0 + k_1 = f(k_0). \tag{42}$$

More generally, the Bellman equation is expressed as

$$v(k) = \max_{k' \in \Gamma(k)} \{u(f(k) - k') + \beta v(k')\}, \tag{43}$$

where

- k is the state variable,
- k' is the control variable, and
- $\Gamma(k)$ is the set of feasible next-period capital stocks (for example, $\Gamma(k) = \{k' : 0 \leq k' \leq f(k)\}$ in the Neoclassical Growth Model).

3.4 The Dynamic Programming Equation and Policy Function

The Bellman equation, also known as the dynamic programming equation (DPE), may also be written as

$$v(k) = \max_{k' \in \Gamma(k)} \{F(k, k') + \beta v(k')\}. \tag{44}$$

If a maximum exists, the policy function (or decision rule) $g(k)$ is defined by

$$g(k) = \text{argmax}_{k' \in \Gamma(k)} \{F(k, k') + \beta v(k')\}. \tag{45}$$

This formulation presumes that the maximum exists so that $g(k)$ is well defined.

3.5 Relationship Between v and g

If the value function $v(k)$ were known, one could determine the policy function by solving

$$\max_{k' \in \Gamma(k)} \left\{ F(k, k') + \beta v(k') \right\}, \tag{46}$$

for each k. Conversely, if the policy function $g(k)$ were known, the value function could be recovered via

$$v(k) = F(k, g(k)) + \beta v(g(k)). \tag{47}$$

In practice, however, neither $v(k)$ nor $g(k)$ is known a priori, and they must be determined simultaneously through the solution of the Bellman equation.

3.6 The T Operator and an Iterative Solution

Define the operator T by

$$Tv(k) = \max_{k' \in \Gamma(k)} \left\{ F(k, k') + \beta v(k') \right\}. \tag{48}$$

An intuitive iterative process for solving for $v(k)$ is as follows:

1. Choose an arbitrary initial function $v_0(k)$ (for example, $v_0(k) = 0$ for all k).
2. For each k, compute

$$v_{n+1}(k) = Tv_n(k) = \max_{k' \in \Gamma(k)} \left\{ F(k, k') + \beta v_n(k') \right\}. \tag{49}$$

Under appropriate conditions, the sequence $\{v_n(k)\}$ converges to the unique fixed point $v(k)$, which is the solution of the Bellman equation.

4 Bellman Equation and Its Iterative Solution

4.1 Bellman Equation Solution

Intuitively, it is possible to find v by the following iterative process. First, one picks any initial function v^0, for example,

$$v^0(k) = 0 \quad \forall\, k.$$

Then, for any value of k, one finds v^{n+1} by evaluating the right-hand side of (13) using v^n. The outcome of this process is a sequence of functions $\{v^j\}_{j=0}^{\infty}$ which converges to v. Moreover, the function v is strictly concave, strictly increasing, and differentiable.

4.2 The Bellman Equation

Mathematically, it remains to ensure that all the operations above are well defined. To this end, some theorems are needed. For example, the Contraction Mapping Theorem (CMT) is used to find the solution V.

5 Remarks on Dynamic Programming

5.1 Optimization and Transversality

In solving the dynamic programming equation (DPE), note that the maximization is infinite-dimensional. Ordinary Kuhn-Tucker methods can be used without reference to additional conditions, such as the transversality condition. However, the reason that a transversality condition is not needed here is subtle and mathematical in nature. In the statements and proofs of equivalence between the sequential and recursive methods, it is necessary to impose conditions on the function v: not any function is allowed.

5.2 Time Consistency and the Principle of Optimality

Decisions made today are not independent of future outcomes. They affect the future via changes in state variables rather than via changes in future decision rules. This time consistency of decision rules is known as the Principle of Optimality. In other words, it states that an optimal plan can be decomposed into two parts: the optimal decision today and the optimal continuation path. Dynamic programming exploits this principle.

6 Relationship between Sequential and Recursive Problems

6.1 Assumptions and Equivalence of Values

Assume that $\Gamma(k)$ is nonempty for all $k \in K$, and that for all $k_0 \in K$ and $\hat{k} \in \Phi(k_0)$,

$$\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(k_t, k_{t+1})$$

exists and is finite. Here, K is the space of k's, $\hat{k}$ is a sequence of k's, and Φ is the set of feasible sequences.

Theorem 1 (Equivalence of Values): *Suppose the assumption holds. Then for any k, any solution $V^{\sim}(k)$ to the sequential problem is also a solution to the recursive problem. Moreover, any solution $V(k)$ to the recursive problem is also a solution to the sequential problem, so that $V^{\sim}(k) = V(k)$ for all k.*

6.2 Principle of Optimality

Theorem 2 (Principle of Optimality): *Suppose the assumption holds. Let there be a feasible plan that attains $V^{\sim}(k_0)$ in the sequential problem. Then*

$$V^{\sim}(k_t) = F(k_t, k_{t+1}) + \beta V^{\sim}(k_{t+1}) \tag{18}$$

for $t = 0, 1, \ldots$ with k_0 given. Moreover, if any $\hat{k} \in \Phi(k_0)$ satisfies this equation then it will be a solution for the sequential problem.

7 Contraction Mapping and Blackwell's Sufficient Conditions

7.1 Contraction Mapping

7.1.1 Definitions

Let (S, d) be a metric space and let $T : S \to S$ be an operator mapping S into itself. If for some $\beta \in (0, 1)$,

$$d(Tz_1, Tz_2) \le \beta d(z_1, z_2) \quad \forall\, z_1, z_2 \in S, \tag{50}$$

then T is a contraction mapping (with modulus β). In other words, a contraction mapping brings elements of the space S uniformly closer to one another. A fixed point of T is any element $z \in S$ satisfying $Tz = z$.

7.1.2 The Contraction Mapping Theorem (CMT)

Theorem (CMT): Let (S, d) be a complete metric space and suppose that $T : S \to S$ is a contraction. Then T has a unique fixed point, $\hat{z}$; that is, there exists a unique $\hat{z} \in S$ such that

$$T\hat{z} = \hat{z}.$$

7.2 Blackwell's Sufficient Conditions

7.2.1 Notation

For a real-valued function $f(\cdot)$ and some constant $c \in \mathbb{R}$, we define

$$(f + c)(x) \equiv f(x) + c.$$

7.2.2 Theorem (Blackwell's Sufficient Conditions for a Contraction)

Let $X \subseteq \mathbb{R}^K$, and let $B(X)$ denote the space of bounded functions $f : X \to \mathbb{R}$ equipped with the sup norm $\| \cdot \|$. Suppose that $B'(X) \subset B(X)$, and let $T : B'(X) \to B'(X)$ be an operator satisfying the following two conditions:

1. **Monotonicity:** For any $f, g \in B'(X)$, if $f(x) \leq g(x)$ for all $x \in X$, then $(Tf)(x) \leq (Tg)(x)$ for all $x \in X$.

2. **Discounting:** There exists $\beta \in (0, 1)$ such that

$$[T(f + c)](x) \leq (Tf)(x) + \beta c$$

for all $f \in B(X)$, $c \geq 0$, and $x \in X$.

Then T is a contraction with modulus β on $B'(X)$.

7.3 Summary of Main Concepts

The recursive solution to the problem is shown to be equivalent to the solution of the sequential problem by means of the *Principle of Optimality.* The value function V, which solves the Bellman equation, is a fixed point of an operator. Under appropriate assumptions, the operator on the right-hand side of the Bellman equation is a contraction mapping. When operators are contractions, repeated application of the operator converges to the fixed point; this is guaranteed by the *Contraction Mapping Theorem.* Finally, the *Blackwell Sufficient Condition* provides criteria for when an operator is a contraction mapping.

8 Methods to Find the Value Function V

There are two principal methods for finding V:

Method 1: Guess and Verify One can guess a functional form for V and then plug it into the right-hand side (RHS) of the Bellman equation to verify that the guess is correct.

Method 2: Iterative Process The value function is obtained by an iterative process, either analytically (as indicated by the relevant theorem) or numerically using a computer.

8.1 Example: Solving the Bellman Equation

8.1.1 Setup of the Example

Let

$$u(c) = \log(c) \quad \text{and} \quad F(k) = Ak^{\alpha}, \quad \delta = 1.$$

Then the Bellman equation reads

$$v(k) = \max_{k' \geq 0} \Big\{ \log[Ak^{\alpha} - k'] + \beta v(k') \Big\}. \tag{19}$$

We apply the iterative process as follows.

8.1.2 Iteration 1

First, guess

$$v^0(k) = 0.$$

Then,

$$v^1(k) = \max_{k' \geq 0} \{\log[Ak^\alpha - k']\}. \tag{20}$$

This expression is maximized when $k' = 0$, so that

$$v^1(k) = \log[Ak^\alpha] = \log(A) + \alpha \log(k). \tag{21}$$

8.1.3 Iteration 2

Next, we obtain

$$v^2(k) = \max_{k' \geq 0} \Big\{ \log[Ak^\alpha - k'] + \beta \big[\log(A) + \alpha \log(k')\big] \Big\}. \tag{22}$$

8.1.4 Finding the Optimum in Iteration 2

Taking the derivative with respect to k':

$$\frac{\partial v^2(k)}{\partial k'} = \frac{-1}{Ak^\alpha - k'} + \frac{\beta\alpha}{k'} = 0. \tag{23}$$

Thus,

$$k' = \frac{\beta\alpha A k^\alpha}{1 + \alpha\beta}. \tag{24}$$

Plugging k' back into the Bellman equation gives:

$$v^2(k) = \log\left[Ak^\alpha - \frac{\beta\alpha A k^\alpha}{1 + \beta\alpha}\right] + \beta\left[\log(A) + \alpha \log\left(\frac{\beta\alpha A k^\alpha}{1 + \alpha\beta}\right)\right]. \tag{25}$$

So,

$$v^2(k) = (\alpha + \alpha^2\beta)\log(k) + \log\left(\frac{A}{1 + \beta\alpha}\right) + \beta\log(A) + \frac{\alpha\beta}{1 + \alpha\beta}\log(\beta\alpha A). \tag{26}$$

8.1.5 Iteration 3 and Functional Form

Notice that $v^1(k) \neq v^2(k)$; however, the solution v appears to have the form

$$\tilde{v}(k) = a\log(k) + b. \tag{27}$$

Using this form, we further iterate:

$$v^3(k) = \max_{k' \geq 0} \Big\{ \log[Ak^\alpha - k'] + \beta[a\log(k') + b] \Big\}. \tag{28}$$

Taking derivatives to find k':

$$\frac{\partial v^3(k)}{\partial k'} = \frac{-1}{Ak^\alpha - k'} + \frac{a\beta}{k'} = 0, \tag{29}$$

which yields

$$k' = \frac{a\beta A k^\alpha}{1 + a\beta}. \tag{30}$$

Subsequently, we obtain

$$v^3(k) = (\alpha + a\alpha\beta)\log(k) + (\text{constant}). \tag{51}$$

To have consistency, we require

$$a = \alpha + a\alpha\beta,$$

and also that the constant term satisfies

$$b = \log(A)(1 + \beta a) + b\beta + b\beta \log \frac{a\beta}{1 + a\beta} + \frac{1}{1 + a\beta}.$$

Thus, solving the two equations for the two unknowns gives

$$a = \frac{\alpha}{1 - \beta\alpha},$$

and

$$b = \frac{1}{1-\beta}\frac{1}{1-\alpha\beta}\Big[\log(A) + (1-\alpha\beta)\log(1-\alpha\beta) + \alpha\beta\log(\alpha\beta)\Big].$$

Returning to the decision rule,

$$k' = \frac{a\beta A k^\alpha}{1 + a\beta}, \tag{31}$$

and substituting the value for a, one finds

$$k' = \alpha\beta A k^\alpha. \tag{32}$$

8.2 Solution Techniques

There are several methods available for solving the Bellman equation:

Guess and Verify: Applicable for problems with closed form solutions.

Policy Function Iteration: Also known as "Howard's Improvement Algorithm".

Value Function Iteration: A numerical approach based on successive approximations.

9 Computing the Value Function: Numerical Methods

9.1 Discretizing the State Space

Suppose we force capital to belong to a grid, i.e., we discretize the domain for the capital stock:

$$k = \{k_1, \ldots, k_m\}. \tag{5}$$

Assume the parameters α, β, δ, and A are known. An algorithm is set up to find V numerically.

9.2 The Value Function Iteration Algorithm

The numerical algorithm is implemented as follows:

1. Define the following:
   - n_k: the number of grid points, determined by the trade-off between speed and precision.
   - K: the lower bound of the state space, chosen slightly above 0 since $K = 0$ is a steady state with zero consumption.

- $\bar{K}$: the upper bound of the state space, chosen slightly above the analytically computed steady state level.
- ϵ: the tolerance for error.

2. Set the grid points $\{K_1, K_2, \ldots, K_{n_k}\}$ (by default, these are equidistant). The value function is stored as a set of n_k points $\{V_i\}_{i=1}^{n_k}$.
3. Set an initial value function $V^0 = \{V_i^0\}_{i=1}^{n_k}$. A trivial initial condition is $V^0 = \{0\}_{i=1}^{n_k}$.
4. For each grid point K_i, update the value function as follows:
   (a) Fix the current capital stock at one of the grid points, K_i, beginning with $i = 1$.
   (b) For each possible choice of next period's capital, K_j, $j = 1, \ldots, n_k$, compute

$$V_{i,j}^1 = u\Big(AF(K_i, 1) + (1-\delta)K_i - K_j\Big) + \beta V_j^0. \tag{52}$$

   If the consumption

$$C = AF(K_i, 1) + (1-\delta)K_i - K_j$$

   is negative, assign a very large negative number to $V_{i,j}^1$ so that K_j is never chosen optimally. Note that V_i^1 is a vector of length n_k, with each element $V_{i,j}^1$ representing the value of the Bellman equation conditional upon the choice of next period's capital K_j.

   (c) For each i, choose the j which gives the highest value among the computed $V_{i,j}^1$. Store this maximum as the ith element of the updated value function V^1, and record the optimal decision as $g_i \in \{1, 2, \ldots, n_k\}$.

   (d) Repeat steps (a) through (c) for each grid point K_i. Once the entire grid has been processed, an updated value and policy function are obtained.

5. Compare V^0 and V^1. Compute the distance d, for example, by the sup norm:

$$d = \max_{i \in \{1,2,\ldots,n_k\}} |V_i^1 - V_i^0|.$$

   If $d \leq \epsilon$, convergence is achieved and the numerical approximation of the optimal value function and policy function is complete. The value function is $V^1 = \{V_i^1\}_{i=1}^{n_k}$ and the optimal decision rule is $g = \{g_i\}_{i=1}^{n_k}$. If $d > \epsilon$, update $V^0 = V^1$ and return to the update step.

9.3 Remarks on the Value Function Iteration

Several practical considerations must be addressed when using the algorithm:

- Ensure that the bounds of the state space are not binding. In particular, verify that the optimal decisions satisfy $g_i \in \{2, 3, \ldots, n_k - 1\}$. If not (i.e., if $g_i = 1$ or $g_i = n_k$ for some i), the bounds are too tight and must be relaxed.
- The tolerance ϵ must be sufficiently small. If a reduction in ϵ leads to substantially different optimal decisions, the initial ϵ might have been too large.
- The number of grid points n_k must be large enough. Increase n_k until the value function and the optimal decision rule become insensitive to further increases.
- If some information about the initial guess is available, use it to form a good starting guess. One approach is to run the algorithm first with a small n_k and then use the obtained optimal decision rule to construct a refined initial guess for a run with a larger n_k.
- Many programming languages have built-in functions to implement the maximization step in (4)(b). For instance, in Matlab, the function `max` is used.

10 Competitive Equilibrium in Dynamic Models

10.1 Goal of the Analysis

In contrast to pure maximization setups where a planner makes all decisions, the focus here is on market economies. The aim is to study which economic arrangement or allocation mechanism is used in a model economy to describe decentralized behavior. The analysis begins with frictionless economies, which serve as a benchmark for further extensions and evaluations.

11 Modern Macro and General Equilibrium

11.1 From Old to Modern Macro

Traditional or *Old Macro* models assume aggregate relationships as primitives. For example, in the typical old-fashioned macro model:

- The goods market is characterized by:
  - Consumption function: $C = C_0 + cY$.
  - Investment function: $I = I_0 - bi$.
  - The goods market identity: $Y = C + I + G$, leading to the IS curve:

$$(1 - c)Y = C_0 + I_0 + G - bi.$$

- The money market is characterized by:
  - Money demand function: $L = L_0 + kY - di$.
  - Money supply function: M/P.

  Together, these yield the LM curve:

$$M/P = L_0 + kY - di.$$

The main features of these models are that aggregate relationships and their parameters (such as the marginal propensity to consume) are assumed to be primitives and remain constant, and any expectations (if present) are given or follow predetermined rules.

11.2 Shortcomings of Old Macro Models

Old macro models suffer from several shortcomings:

- They often overlook individual constraints, such as the budget constraint of consumers.
- Agents are assumed not to react to changes in government policies, even though consumption decisions (and thus savings) would vary if the government ran systematic deficits or if the social security system were expected to be insolvent (cf. the Lucas Critique).
- They generally lack consistency between agents' expectations and their actions.

11.3 Modern Macro

Modern macroeconomics adopts a bottom-up approach:

- **Main Features:** The analysis is carried out on an artificial economy, an abstract mathematical construct or "lab", where agents interact in markets and aggregates emerge from individual decisions.
- **Primitives:** The analysis starts with a detailed description of the environment, including the agents' demographics, preferences, endowments, technologies, and the markets in which they can trade.

11.4 Agent Behavior in Modern Macro

Agents are assumed to behave optimally by solving their individual optimization problems. Specifically, each agent maximizes utility subject to a budget constraint and forms rational expectations by using all available information to make the best possible forecasts.

12 Competitive Equilibrium

12.1 Definition and Concept

A competitive equilibrium is defined as an allocation (i.e., a vector of quantities) and a price system (i.e., a vector of prices) such that:

1. The quantities are the solutions of all agents' optimization problems, given the prices.

2. Markets clear, meaning that supply equals demand in $n-1$ markets.

In such an equilibrium, individual households and firms interact in markets, and aggregate consistency is achieved when the quantities supplied and demanded are equal for each market.

13 Competitive Equilibrium: Framework and Setup

This section outlines the steps required to set up a competitive equilibrium. The process involves describing the environment, solving the individual agents' problems, stating the market clearing conditions, and finally formally defining the equilibrium.

13.1 Steps to Set Up a Competitive Equilibrium

The establishment of a competitive equilibrium proceeds as follows:

Step 1: **Describe the environment:** Specify the agents (consumers, firms, government, etc.), their preferences and endowments (wealth, time, etc.), the available technologies (for production, human capital, etc.), and the markets in which they interact.

Step 2: **Solve each agent's problem:** Write down and solve the optimization problem for each agent.

Step 3: **State market clearing conditions:** For each market, aggregate individual supply and demand so that aggregate supply equals aggregate demand. (Recall Walras's law: if $n-1$ markets clear, the nth market will clear as well.)

Step 4: **Formally define the equilibrium:** Collect all endogenous variables and equations (first-order conditions, value functions, policy functions, and market clearing conditions) to characterize the equilibrium allocation (quantities) and prices.

13.2 Detailed Environment Description

13.2.1 Step 1: Describe the Environment of the Economy

The environment is described by listing the agents and outlining their characteristics. This involves specifying:

- The agents and their characteristics (e.g., whether they are equal or differ in age).

- The behavior of the agents (maximizing utility for consumers, maximizing profits for firms, etc.).

- The preferences and endowments (wealth, time, etc.).
- The technologies available (to produce output, accumulate human capital, etc.).
- The markets in which these agents interact.

13.2.2 Step 2: Solving the Agents' Problems

Once the environment is defined, the next step is to write down the optimization problem for each agent and derive their decision rules that determine their choices.

13.2.3 Step 3: Market Clearing Conditions

Market clearing is achieved by aggregating individual supplies and demands. In each market, the aggregate supply and aggregate demand are determined, for instance:

$$\text{Aggregate Supply} = \sum \text{individual supply}.$$

For market i, the market clearing condition is:

$$\text{Aggregate Supply in market } i = \text{Aggregate Demand in market } i.$$

Recall Walras's Law: in an economy with n markets, if $n-1$ markets clear, the final market will clear automatically.

13.2.4 Step 4: Defining the Competitive Equilibrium

The competitive equilibrium is defined by collecting all the endogenous variables (for example, consumption choices and prices) and the equations (first-order conditions, value functions, policy functions, and market clearing conditions). With these elements, the equilibrium is characterized by the solution for the allocation (quantities) and prices.

14 Economic Mechanisms in the Competitive Equilibrium

This section explains the workings of the economy by describing the behavior of households and firms, as well as the role of market clearing conditions.

14.1 Households and Firms in the Economy

Households maximize their utility by choosing consumption quantities given their wealth, which is determined by factor ownership valued at given prices. They take prices as parameters and face a budget constraint defined by the maximum monetary value of goods they can purchase. Moreover, the chosen quantities must be feasible—that is, the aggregate demand must be producible with the available technology and factor supplies, which are themselves determined by factor remuneration (prices).

Firms, on the other hand, choose the production volume that maximizes profits at the given prices. The framework accommodates different economic arrangements that lead to the same final allocations (for example, whether firms rent inputs each period or own long-lived capital).

14.2 Advantages of the Modern Macro Approach

The modern approach to macroeconomics and competitive equilibrium offers several gains:

(a) Consistency: Aggregate relationships constructed from individual behaviors automatically satisfy individual constraints. For instance, an aggregate consumption function derived from household behavior cannot violate any individual's budget constraint.

(a) **Transparency:** The underlying assumptions about the fundamentals of the economy are stated clearly.

(a) **No Arbitrary Behavior:** In older macroeconomic models, behavior was often assumed arbitrarily, whereas in modern macroeconomics, behavior is derived from the optimization problems of agents.

(a) **Endogenous Expectations:** Expectations are derived within the model and are consistent with the overall behavior of the economy.

(a) **Welfare Analysis:** Changes in utility due to policy changes can be studied directly.

(a) **Testing:** The model outcomes can be compared with both macro and micro data.

14.3 Limitations

It is important to note that modern market economies may exhibit frictions (such as incomplete information, externalities, and market power) that are not well captured by models assuming frictionless markets. Nonetheless, for a frictionless economy, competitive equilibrium analysis remains both suitable and highly useful.

15 An Illustrative Example: A Static Economy

This section presents a static, one-period economy with identical households where households receive endowments that they directly consume.

15.1 Step 1: Environment Setup

Demographics and Preferences. There are N households that live for one period in an endowment economy with no firms. Households value the consumption of two different goods according to a utility function $u(c_1, c_2)$.

Endowments and Technology. Each household receives an endowment (e_1, e_2). There is no production or storage technology; endowments cannot be stored. Resource constraints are given by:

$$Ne_1 = Nc_1 \quad \text{and} \quad Ne_2 = Nc_2.$$

Markets. Competitive markets exist for the two goods, with two prices p_1 and p_2 defined. (One price can be normalized, for example, $p_1 = 1$.)

15.2 Step 2: The Household Problem

Households maximize the utility function $u(c_1, c_2)$ subject to the budget constraint. Taking prices p_1 (normalized) and p_2 (with $p = p_2/p_1$) as given along with endowments e_1 and e_2, the household solves:

$$\max_{c_1,c_2} u(c_1, c_2) \quad \text{s.to} \quad c_1 + p\,c_2 = e_1 + p\,e_2.$$

A solution for the household problem is a pair (c_1, c_2).

Method of Solution. One method involves setting up the Lagrangian:

$$L = u(c_1, c_2) + \lambda\,[e_1 + p\,e_2 - c_1 - p\,c_2]\,,$$

while an alternative is to substitute and write:

$$\max u(e_1 + p\,e_2 - p\,c_2, c_2).$$

First-Order Conditions. The first-order conditions (FOCs) for c_1 and c_2 are:

$$\frac{\partial L}{\partial c_i} = u_i(c_1, c_2) - \lambda p_i = 0, \quad i = 1, 2,$$

where λ is the marginal utility of income. Taking the ratio of the FOCs yields:

$$\frac{u_2}{u_1} = p,$$

so that the marginal rate of substitution equals the relative price. Finally, assuming a specific utility function, for example,

$$u(c_1, c_2) = \ln(c_1) + \beta \ln(c_2),$$

leads to the solution:

$$\frac{u_2}{u_1} = \beta \frac{c_1}{c_2} = p,$$

and substituting back into the budget constraint yields:

$$c_1 + \beta c_1 = W = e_1 + p\, e_2, \quad \text{so that} \quad c_1 = \frac{W}{1+\beta}, \quad c_2 = \frac{\beta W}{1+\beta}.$$

15.3 Step 3: Market Clearing

Market Clearing Conditions. In this economy there are two markets. In each market, households supply e_i and demand c_i. Market clearing requires:

$$\text{Aggregate Demand} = \text{Aggregate Supply}.$$

More specifically, for each good i:

$$D_i(p, e_1, e_2) = \sum_{j=1}^{N} c_i = N c_i(p, e_1, e_2), \quad \text{and} \quad S_i = \sum_{j=1}^{N} e_i = N e_i.$$

Thus, the market-clearing condition is:

$$c_i = e_i.$$

15.4 Step 4: Defining the Competitive Equilibrium

A competitive equilibrium in this static economy is an allocation (c_1, c_2) and a price p such that:

1. The pair (c_1, c_2) solves the household maximization problem (satisfying the two FOCs and the budget constraint) given p.
2. The markets clear.

There are $2N + 1$ endogenous variables (individual consumption and price), with $2N$ optimality conditions and 2 market clearing conditions.

16 Dynamic Economies

This section introduces dynamic economies and discusses the nature of trade over time.

16.1 Transition from Static to Dynamic Settings

In dynamic economic setups, it is necessary to specify how trade takes place over time—whether through assets (and which types) or through other sequential mechanisms. Equilibria in dynamic models can be defined either as infinite sequences or recursively using functions.

16.2 An Endowment Economy with Date-0 Trade

Environment. Consider an economy with a single consumer with an infinite life. This is an exchange economy (with no production) where at each date t the consumer is endowed with $w_t \in \mathbb{R}$ units of the consumption good. Since there is no storage technology, the consumer must consume all his endowment in each period.

Market Structure and Budget Constraint. The commodities are the consumption goods at different dates, each with its own price p_t. Normalizing $p_0 = 1$, the price of consumption at t is given relative to time 0. Then the following expressions represent the value of the endowment and the expenditure:

$$\text{Value of the endowment:} \quad \sum_{t=0}^{\infty} p_t w_t \tag{1}$$

$$\text{Value of expenditure:} \quad \sum_{t=0}^{\infty} p_t c_t \tag{2}$$

The budget constraint is:

$$\sum_{t=0}^{\infty} p_t c_t \leq \sum_{t=0}^{\infty} p_t w_t \tag{3}$$

This structure assumes that purchases and sales of consumption goods for every period are carried out at $t = 0$. This market is known as an Arrow-Debreu-McKenzie or date-0 market.

Definition of Competitive Equilibrium. A competitive equilibrium in this setting is a vector of prices $\{p_t\}_{t=0}^{\infty}$ and a vector of quantities $\{c_t\}_{t=0}^{\infty}$ such that:

1. The sequence $\{c_t\}_{t=0}^{\infty}$ solves the consumer's maximization problem:

$$\{c_t\}_{t=0}^{\infty} = \arg \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

   subject to

$$\sum_{t=0}^{\infty} p_t c_t \leq \sum_{t=0}^{\infty} p_t w_t, \quad c_t \geq 0 \ \forall t.$$

2. Market clearing requires that

$$c_t = w_t.$$

Equilibrium Prices. From the first-order conditions (FOCs) for the consumer's problem, we have:

$$\beta^t u'(w_t) = \lambda p_t \quad \forall t,$$

which implies that

$$\frac{p_t}{p_{t+1}} = \frac{1}{\beta} \frac{u'(w_t)}{u'(w_{t+1})}.$$

Thus, the relative price of today's consumption in terms of tomorrow's consumption equals the discounted ratio of marginal utilities.

16.3 An Endowment Economy with Sequential Trade

Market Structure. Consider the same economy but now with sequential trade, which allows for one-period loans with a gross interest rate $R_t \equiv 1 + r_t$. Let a_t denote the net asset position (net savings) of the agent at time t. In this setup, assets allow the agent to transfer wealth from one period to the next. With a single agent, the asset market clears only if:

$$a_t = 0 \quad \forall t.$$

Period-by-Period Budget Constraint. In period t, the consumer faces:

$$c_t + a_{t+1} = a_t R_t + w_t.$$

Definition of Competitive Equilibrium. A competitive equilibrium in the sequential trade framework is a set of sequences $\{c_t\}_{t=0}^{\infty}$, $\{a_{t+1}\}_{t=0}^{\infty}$, and $\{R_t\}_{t=0}^{\infty}$ such that:

1. The sequence $\{c_t, a_{t+1}\}_{t=0}^{\infty}$ solves the consumer's maximization problem:

$$\{c_t, a_{t+1}\}_{t=0}^{\infty} = \arg \max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

   subject to

$$c_t + a_{t+1} = a_t R_t + w_t \quad \forall t, \quad c_t \geq 0 \ \forall t, \quad a_0 = 0,$$

   and the non-Ponzi game condition

$$\lim_{t \to \infty} a_{t+1} \left(\prod_{s=0}^{t} R_{s+1} \right)^{-1} = 0.$$

2. The asset market clearing condition is

$$a_t = 0 \quad \forall t.$$

3. Goods market clearing requires

$$c_t = w_t.$$

Equilibrium Interest Rate. From the consumer's Euler equation derived from the FOCs:

$$u'(w_t) = \beta u'(w_{t+1}) R_{t+1},$$

which implies that

$$R_{t+1} = \frac{1}{\beta} \frac{u'(w_t)}{u'(w_{t+1})}.$$

17 Neoclassical Growth Models

This section extends the analysis to a production economy in which households and firms interact in a dynamic setting. Two market structures are discussed: Date-0 (Arrow-Debreu) trade and sequential trading.

17.1 Neoclassical Growth Model with Date-0 Trade

17.1.1 Assumptions and Setup

The key assumptions and setup for the Date-0 trade model are as follows:

- The consumer is endowed with 1 unit of time each period which can be allocated between labor and leisure.
- The utility from the consumption and leisure stream is given by:

$$U\left(\{c_t, 1-n_t\}_{t=0}^{\infty}\right) = \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

  (Here, leisure is not valued; equivalently, labor supply bears no utility cost. It is assumed that $u(c)$ is strictly increasing and strictly concave.)

- The consumer owns capital and rents it to firms in exchange for a rental rate r_t. Capital depreciates at rate δ.
- The consumer supplies labor at wage rate w_t.
- The production function for the consumption/investment good is $F(K, n)$, which is strictly increasing, concave, and homogeneous of degree 1.

  Prices are defined as follows:

- p_t: price of consumption goods at time t, normalized such that $p_0 = 1$.
- $p_t r_t$: price of capital services at time t.
- $p_t w_t$: price of labor at time t.

17.1.2 Definition of the Competitive Equilibrium

A competitive equilibrium (CE) is defined by sequences of prices and quantities:

Prices: $\{p_t\}_{t=0}^{\infty}$, $\{r_t\}_{t=0}^{\infty}$, $\{w_t\}_{t=0}^{\infty}$

Quantities: $\{c_t\}_{t=0}^{\infty}$, $\{K_{t+1}\}_{t=0}^{\infty}$, $\{n_t\}_{t=0}^{\infty}$

such that:

1. The sequences $\{c_t\}_{t=0}^{\infty}$, $\{K_{t+1}\}_{t=0}^{\infty}$, and $\{n_t\}_{t=0}^{\infty}$ solve the consumer's problem:

$$\{c_t, K_{t+1}, n_t\}_{t=0}^{\infty} = \arg \max_{\{c_t, K_{t+1}, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

   subject to:

$$\sum_{t=0}^{\infty} p_t \left[c_t + K_{t+1}\right] = \sum_{t=0}^{\infty} p_t \left[r_t K_t + (1-\delta)K_t + n_t w_t\right],$$

   with $c_t \geq 0$ for all t and K_0 given. Notice that since labor has no utility cost and $w_t > 0$, the consumer supplies all available labor (i.e., $n_t = 1$ for all t).

2. The firm's problem for each period t is:

$$(K_t, n_t) = \arg \max_{K_t, n_t} \left\{p_t F(K_t, n_t) - p_t r_t K_t - p_t w_t n_t\right\}.$$

   This can be expressed equivalently as:

$$r_t = F_K(K_t, 1) \quad \text{and} \quad w_t = F_n(K_t, 1).$$

3. Feasibility (market clearing) requires that:

$$c_t + K_{t+1} = F(K_t, 1) + (1-\delta)K_t. \tag{2}$$

17.1.3 Characterization via First-Order Conditions

For the consumer's problem, differentiating with respect to c_t yields:

$$\beta^t u'(c_t) = p_t \lambda, \tag{3}$$

$$\beta^{t+1} u'(c_{t+1}) = p_{t+1} \lambda. \tag{4}$$

Combining (3) and (4) gives:

$$\frac{p_t}{p_{t+1}} = \frac{1}{\beta} \frac{u'(c_t)}{u'(c_{t+1})}. \tag{5}$$

Differentiating with respect to K_{t+1} results in:

$$p_t \lambda = p_{t+1} \lambda \left[r_{t+1} + (1 - \delta) \right], \tag{6}$$

so that:

$$\frac{p_t}{p_{t+1}} = r_{t+1} + 1 - \delta, \tag{7}$$

and using the firm equilibrium condition:

$$\frac{p_t}{p_{t+1}} = F_K(K_{t+1}, 1) + 1 - \delta. \tag{8}$$

Combining (5) and (8) results in the intertemporal Euler equation:

$$u'(c_t) = \beta u'(c_{t+1}) \left[F_K(K_{t+1}, 1) + 1 - \delta \right], \tag{9}$$

which is exactly the Euler equation from the planner's problem. This confirms that the competitive equilibrium allocation is Pareto optimal (First Welfare Theorem).

17.2 Neoclassical Growth Model with Sequential Trading

17.2.1 Assumptions and Market Structure

In the sequential trading framework, prices are defined as follows:

- R_t: rental rate (price of capital services at time t). For variety, we define $R_t = r_t + 1 - \delta$, which is the return on capital net of depreciation.
- w_t: wage rate (price of labor at time t).

17.2.2 Definition of the Competitive Equilibrium

A competitive equilibrium in this sequential trading framework is a set of sequences:

Prices: $\{R_t\}_{t=0}^{\infty}$, $\{w_t\}_{t=0}^{\infty}$

Quantities: $\{c_t\}_{t=0}^{\infty}$, $\{K_{t+1}\}_{t=0}^{\infty}$, $\{n_t\}_{t=0}^{\infty}$

such that:

1. The sequences $\{c_t, K_{t+1}, n_t\}_{t=0}^{\infty}$ solve the consumer's problem:

$$\{c_t, K_{t+1}, n_t\}_{t=0}^{\infty} = \arg \max_{\{c_t, K_{t+1}, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to:

$$c_t + K_{t+1} = R_t K_t + n_t w_t, \quad c_t \geq 0 \ \forall t, \quad K_0 \text{ given},$$

and the appropriate non-Ponzi game condition.

2. The firm's problem for each period is:

$$(K_t, 1) = \arg\max_{K_t, n_t} \left\{ F(K_t, n_t) - R_t K_t + (1-\delta)K_t - w_t n_t \right\}.$$

3. Feasibility (market clearing) is given by:

$$c_t + K_{t+1} = F(K_t, 1) + (1-\delta)K_t. \tag{10}$$

17.2.3 Characterization via First-Order Conditions

For the consumer, differentiating with respect to c_t gives:

$$\beta^t u'(c_t) = \beta^t \lambda_t, \tag{11}$$

$$\beta^{t+1} u'(c_{t+1}) = \beta^{t+1} \lambda_{t+1}, \tag{12}$$

so that:

$$\frac{\lambda_t}{\lambda_{t+1}} = \frac{u'(c_t)}{u'(c_{t+1})}. \tag{13}$$

Differentiating with respect to K_{t+1} yields:

$$\beta^t \lambda_t = \beta^{t+1} \lambda_{t+1} R_{t+1}, \tag{14}$$

implying:

$$\frac{\lambda_t}{\lambda_{t+1}} = \beta R_{t+1}. \tag{15}$$

Combining (13) and (15) results in:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta R_{t+1}, \tag{16}$$

and using the firm's equilibrium condition:

$$R_{t+1} = F_K(K_t, 1) + (1-\delta), \tag{17}$$

we obtain:

$$u'(c_t) = \beta u'(c_{t+1}) \left[F_K(K_{t+1}, 1) + 1 - \delta \right]. \tag{18}$$

This is identical to the planner's Euler equation, confirming that both the Arrow-Debreu-McKenzie (date-0) and the sequential market equilibria are Pareto optimal.

18 Conclusion and Game Plan for Further Studies

In summary, these notes have examined competitive equilibrium in both static and dynamic environments. The material has covered the basic steps to set up a competitive equilibrium, including household and firm optimization, market clearing, and equilibrium definitions in a static economy; it then extended the discussion to dynamic settings, featuring both date-0 (Arrow-Debreu) and sequential trading frameworks. Finally, the neoclassical growth model was analyzed under both market structures and characterized via first-order conditions.

Game Plan for Further Studies: Future studies should include:

- A detailed examination of competitive equilibrium in endowment economies with both date-0 trade and sequential trade.
- An analysis of competitive equilibrium in production economies using the Neoclassical Growth Model (NGM) under both trade arrangements, including a recursive formulation of the competitive equilibrium.

19 Recursive Competitive Equilibrium: Introduction

The concept of a recursive competitive equilibrium (RCE) is characterized by the absence of date-0 trading and the requirement of sequential trading. Instead of representing economic decisions with sequences or vectors, an RCE is described by a set of functions—quantities, utility levels, and prices—that depend on the "state," that is, the relevant initial condition. In a manner analogous to dynamic programming, these functions specify the evolution of the economy for every individual consumer given an arbitrary initial state.

19.1 Planner's Problem and the Decentralized Formulation

First, recall the planner's problem:

$$V(K) = \max_{c,\, K' \geq 0} \{u(c) + \beta V(K')\} \tag{1}$$

subject to

$$c + K' = F(K, 1) + (1 - \delta)K. \tag{2}$$

In the sequential formulation of the decentralized problem, sequences of factor remunerations in equilibrium are given by:

$$R_t = F_K(K_t, 1) + (1 - \delta) \tag{3}$$

$$w_t = F_n(K_t, 1), \tag{4}$$

where both expressions are functions of the aggregate level of capital.

19.2 Price Laws and Budget Constraints in the Recursive Framework

In dynamic programming terms, there is a law of motion for the prices of factors expressed as functions of aggregate capital:

$$R = R(\bar{K}) \quad \text{and} \quad w = w(\bar{K}), \tag{5}$$

where $\bar{K}$ denotes the aggregate capital in the economy. Consequently, the budget constraint in the decentralized dynamic programming problem becomes:

$$c + K' = R(\bar{K})K + w(\bar{K}). \tag{6}$$

19.3 Dynamic Programming Problem of the Individual Agent

Given that the two variables important for decision-making are K and $\bar{K}$, the individual's dynamic programming problem can be written as:

$$V(K, \bar{K}) = \max_{c,\, K' \geq 0} \{u(c) + \beta V(K', \bar{K}')\} \tag{7}$$

subject to

$$c + K' = F(K, 1) + (1 - \delta)K. \tag{8}$$

The agent must also specify $\bar{K}'$, which is determined by the agent's perceived law of motion for aggregate capital. It is assumed that the agent's perception is correct and corresponds to the actual law of motion:

$$\bar{K}' = G(\bar{K}),$$

where G represents the outcome of the economy's equilibrium capital accumulation decisions.

19.4 The Recursive Problem

With all the necessary elements, the consumer's dynamic problem is formulated as:

$$V(K, \bar{K}) = \max_{c, K' \geq 0} \{u(c) + \beta V(K', \bar{K}')\} \tag{9}$$

subject to

$$c + K' = R(\bar{K})K + w(\bar{K}), \tag{10}$$

$$\bar{K}' = G(\bar{K}). \tag{11}$$

The solution yields a policy function for the individual's law of motion for capital:

$$K' = g(K, \bar{K}) = \arg \max_{K' \in [0, R(\bar{K})K + w(\bar{K})]} \{u[R(\bar{K})K + w(\bar{K}) - \bar{K}'] + \beta V(K', \bar{K}')\} \tag{12}$$

subject to

$$\bar{K}' = G(\bar{K}). \tag{13}$$

19.5 Definition of a Recursive Competitive Equilibrium

A *competitive equilibrium* in the recursive framework is defined as a set of functions:

$$\text{Quantities: } G(\bar{K}), \quad g(K, \bar{K});$$

$$\text{Utility Level: } V(K, \bar{K});$$

$$\text{Prices: } R(\bar{K}), \quad w(\bar{K}),$$

such that:

(1) $V(K, \bar{K})$ solves the consumer problem and $g(K, \bar{K})$ is the corresponding policy function.

(2) Prices are determined competitively:

$$R(\bar{K}) = F_K(\bar{K}, 1) + 1 - \delta, \tag{14}$$

$$w(\bar{K}) = F_n(\bar{K}, 1). \tag{15}$$

(3) Consistency is satisfied:

$$G(\bar{K}) = g(\bar{K}, \bar{K}) \quad \forall \bar{K}, \tag{16}$$

(4) Market clearing holds:

$$\bar{C} + \bar{K}' = F(\bar{K}, 1) + (1 - \delta)\bar{K}.$$

19.6 Additional Comments on Consistency

The term "consistency" emphasizes that the aggregate law of motion perceived by the agent must align with the actual behavior of the individual agents. In the recursive framework, consistency corresponds to the idea in the sequential framework that the sequences chosen by consumers, such as capital holdings, must satisfy the first-order conditions given prices that are determined from firms' first-order conditions evaluated using the same capital sequences.

19.7 Example 1: NGM with Leisure

Consider a variation of the one-sector growth model that incorporates labor choice, where households derive disutility from working. In this model, the individual state is the household's asset level, a. For prices to be determined, both aggregate capital and aggregate labor must be known; however, aggregate labor is an aggregation of individual labor supply which depends solely on capital. Therefore, aggregate capital, K, serves as the aggregate state. In this context, there are two perceived laws of motion:

$$K' = G(K) \quad \text{and} \quad N = H(K).$$

19.8 Example 2: NGM with Leisure and Taxes

This example considers an economy similar to Example 1 but introduces proportional taxes that are rebated to consumers in a lump sum. All objects are identical to those in the previous economy, except that households now take transfers as given by:

$$TR(K, H(K)) = \tau F_2(K, H(K))H(K). \tag{53}$$

The household's problem is formulated as:

$$V(a, K) = \max_{c,\, a',\, n} \{u(c, 1-n) + \beta V(a', K')\} \tag{54}$$

subject to

$$c + a' = w(K, H(K))n(1-\tau) + (1 + r(K, H(K)))a + TR(K, H(K)), \tag{55}$$

$$TR(K, H(K)) = \tau F_2(K, H(K))H(K), \tag{56}$$

$$K' = G(K), \quad N = H(K). \tag{57}$$

19.9 Example 3: NGM with Capital Accumulation by Firms

In this example, firms are responsible for capital accumulation decisions, and the (single) consumer owns stock in these firms. The functions that enter the dynamic programs of both the consumer and the firm are:

$\bar{K}' = G(\bar{K})$: the aggregate law of motion of capital.

$q(\bar{K})$: the current price of next period's consumption.

$V_c(a, \bar{K})$: the consumer's indirect utility as a function of $\bar{K}$ and a.

$a' = g_c(a, \bar{K})$: the policy rule associated with $V_c(a, \bar{K})$.

$V_f(K, \bar{K})$: the market value of a firm as a function of K and aggregate capital $\bar{K}$.

$K' = g_f(K, \bar{K})$: the policy rule associated with $V_f(K, \bar{K})$.

19.9.1 Consumer's Problem

The consumer's dynamic problem is set up as:

$$V_c(a, \bar{K}) = \max_{c \geq 0,\, a'} \{u(c) + \beta V_c(a', \bar{K}')\} \tag{58}$$

subject to

$$c + q(\bar{K})a' = a, \tag{59}$$

$$\bar{K}' = G(\bar{K}). \tag{60}$$

The solution to this problem yields the policy function:

$$a' = g_c(a, \bar{K}). \tag{14}$$

19.9.2 Firm's Problem

The firm's problem is given by:

$$V_f(K, \bar{K}) = \max_{K'}\{F(K, 1) + (1 - \delta)K - K' + q(\bar{K})V_f(K', \bar{K}')\} \tag{61}$$

subject to

$$\bar{K}' = G(\bar{K}). \tag{62}$$

Its solution provides the policy function:

$$K' = g_f(K, \bar{K}).$$

19.9.3 Definition of the Recursive Competitive Equilibrium (RCE)

A recursive competitive equilibrium in this context is defined as a set of functions:

$$\text{Quantities: } G(\bar{K}), \quad g_c(a, \bar{K}), \quad g_f(K, \bar{K});$$

$$\text{Utility Levels: } V_c(a, \bar{K}), \quad V_f(K, \bar{K});$$

$$\text{Prices: } q(\bar{K}),$$

such that:

(1) $V_c(a, \bar{K})$ solves the consumer's problem and $g_c(a, \bar{K})$ is the associated policy function.

(2) $V_f(K, \bar{K})$ solves the firm's problem and $g_f(K, \bar{K})$ is the associated policy function.

(3) Consistency 1:

$$G(\bar{K}) = g_f(\bar{K}, \bar{K}) \quad \forall \bar{K}. \tag{15}$$

(4) Consistency 2:

$$V_f[G(\bar{K}), G(\bar{K})] = g_c[V_f(\bar{K}, \bar{K}), \bar{K}] \quad \forall \bar{K}. \tag{16}$$

19.9.4 Discussion of Consistency Condition 2

The origin of Consistency 2 lies in the fact that if the consumer owns the entire firm, then the ownership must persist into the next period. In other words, if

$$a = V_f(K, \bar{K}), \tag{17}$$

then the value of the firm in the next period is $V_f(K', \bar{K}')$, where

$$V_f(K', \bar{K}') = V_f(g_f(K, \bar{K}), G(\bar{K})). \tag{63}$$

Since the consumer owns the firm next period as well, it follows that

$$a' = V_f(g_f(K, \bar{K}), G(\bar{K})). \tag{18}$$

Recognizing that a' is determined by the policy rule, we have:

$$g_c(a, \bar{K}) = V_f(g_f(K, \bar{K}), G(\bar{K})). \tag{19}$$

Replacing a by combining (17) and (18) gives:

$$g_c(V_f(K, \bar{K}), \bar{K}) = V_f(g_f(K, \bar{K}), G(\bar{K})). \tag{20}$$

Finally, by imposing $K = \bar{K}$ and using Consistency 1, we obtain:

$$g_c(V_f(\bar{K}, \bar{K}), \bar{K}) = V_f(G(\bar{K}), G(\bar{K})). \tag{21}$$

19.10 Optimization under Uncertainty

Consider a two-period economy in which agents consume and save in period 0 and consume and work in period 1. In period 1 the wage rate is uncertain. Suppose there are n possible states of the world, so that $\omega_2 \in \{\omega_1, \ldots, \omega_n\}$, and let

$$\pi_i \equiv Pr(\omega_2 = \omega_i), \quad \text{for } i = 1, \ldots, n.$$

The consumer is an expected utility maximizer with von Neumann-Morgenstern preferences and values leisure in the second period. The utility function is defined as:

$$U = \sum_{i=1}^{n} \pi_i u(c_0, c_{1i}, n_i) \equiv E[u(c_0, c_{1i}, n_i)], \tag{64}$$

and is assumed to take the form

$$U = u(c_0) + \beta \sum_{i=1}^{n} \pi_i [u(c_{1i}) - v(n_i)], \tag{65}$$

with the property that $v'(n_i) > 0$.

19.11 Incomplete Markets

Assume there exists a "risk free" asset, denoted by a, and priced at q, such that every unit of a purchased in period 0 yields 1 unit in period 1 regardless of the state of the world. In period 0 the budget constraint is:

$$c_0 + aq = I. \tag{66}$$

In each state i in period 1 the budget constraint becomes:

$$c_{1i} = a + \omega_i n_i, \quad \text{for } i = 1, \ldots, n. \tag{67}$$

The consumer's maximization problem is given by:

$$\max_{c_0,\, a,\, \{c_{1i}, n_{1i}\}_{i=1}^{n}} \; u(c_0) + \beta \sum_{i=1}^{n} \pi_i \, u(c_{1i}) - \beta \sum_{i=1}^{n} \pi_i \, v(n_i), \tag{68}$$

subject to

$$c_0 + aq = I, \tag{69}$$

$$c_{1i} = a + \omega_i n_i, \quad \text{for } i = 1, \ldots, n. \tag{70}$$

The first-order conditions (FOCs) are derived as follows:

$$\text{For } c_0: \quad u'(c_0) = \lambda, \tag{71}$$

$$\text{Alternatively, } c_0: \quad u'(c_0) = \sum_{i=1}^{n} \lambda_i R, \quad \text{where } R \equiv 1/q, \tag{72}$$

$$\text{For } c_{1i}: \quad \beta \pi_i u'(c_{1i}) = \lambda_i, \tag{73}$$

$$\text{For } n_{1i}: \quad -\beta \pi_i v'(n_{1i}) = \lambda_i \omega_i. \tag{74}$$

Combining the conditions for c_{1i} and n_{1i} yields:

$$-u'(c_{1i})\, \omega_i = v'(n_{1i}), \tag{75}$$

and, for period 0 consumption,

$$u'(c_0) = \beta \sum_{i=1}^{n} \pi_i u'(c_{1i}) R \equiv \beta E[u'(c_{1i}) R]. \tag{76}$$

Thus, the consumer's marginal utility from period 0 consumption is equated to the discounted expected marginal utility from receiving R units in period 1.

20 Introduction to Complete Markets

Instead of a single risk free asset, suppose that "Arrow securities" (state-contingent claims) are traded in period 0. There are n assets, and each unit of asset i pays 1 unit in period 1 if state i is realized and 0 otherwise. The period 0 budget constraint is now:

$$c_0 + \sum_{i=1}^{n} q_i a_i = I. \tag{77}$$

In period 1, if state i is realized, the budget constraint becomes:

$$c_{1i} = a_i + \omega_i n_i. \tag{78}$$

Note that a risk free asset can be synthesized by purchasing one unit of each Arrow security, so that the total price satisfies:

$$q = \sum_{i=1}^{n} q_i. \tag{79}$$

This market structure allows the consumer not only to reallocate income between periods but also across states of the world. In effect, the state-specific wealth transfer can lead to welfare improvements under uncertainty when preferences exhibit risk aversion.

After substituting the expression for a_i from the period-1 constraint into the period-0 constraint, we obtain the consolidated budget constraint:

$$c_0 + \sum_{i=1}^{n} q_i c_{1i} = I + \sum_{i=1}^{n} q_i \omega_i n_i. \tag{80}$$

The first-order conditions are:

$$\text{For } c_0: \quad u'(c_0) = \lambda, \tag{81}$$

$$\text{For } c_{1i}: \quad \beta \pi_i u'(c_{1i}) = \lambda q_i, \tag{82}$$

$$\text{For } n_{1i}: \quad \beta \pi_i v'(n_{1i}) = -q_i \lambda \omega_i. \tag{83}$$

Combining the conditions for c_{1i} and n_{1i}, we obtain:

$$-u'(c_{1i})\, \omega_i = v'(n_{1i}), \tag{84}$$

and

$$u'(c_0) = \frac{\beta \pi_i}{q_i} u'(c_{1i}), \quad i = 1, \ldots, n. \tag{85}$$

Thus, the intra-state consumption-leisure trade-off remains identical to the incomplete markets case, while the additional condition reflects the consumer's ability to allocate consumption across states.

Under this equilibrium allocation, the marginal rate of substitution (MRS) between period 0 consumption and period 1 consumption in state i is given by:

$$\text{MRS}(c_0, c_{1i}) = q_i, \tag{86}$$

and the MRS across states i and j is:

$$\text{MRS}(c_{1i}, c_{1j}) = \frac{q_i}{q_j}. \tag{87}$$

21 Uncertainty in an Endowment Economy

Consider an economy similar to the previous setup but with random endowments. In each period, one of two agents receives an endowment of 0 while the other receives 2. The assignment is random, with the event $s_t = 1$ indicating that agent 1 receives the high endowment and $s_t = 2$ indicating that agent 2 does. The endowments are given by:

$$e_t^1(s_t) = \begin{cases} 2 & \text{if } s_t = 1, \\ 0 & \text{if } s_t = 2, \end{cases} \tag{1}$$

$$e_t^2(s_t) = \begin{cases} 0 & \text{if } s_t = 1, \\ 2 & \text{if } s_t = 2, \end{cases} \tag{2}$$

so that in period t, endowments depend solely on the current event s_t, not on the entire history.

If, for instance, the event history in period 2 is $s_2 = (1, 1, 2)$, then $\pi_t(s_2)$ represents the probability that agent 1 has the high endowment in periods $t = 0$ and $t = 1$, and agent 2 has the high endowment in period 2. Commodities in the economy are now indexed by both time and event histories:

$$(c^1, c^2) = \{c_t^1(s_t),\, c_t^2(s_t)\}_{t=0,\, s_t \in S_t}^{\infty}. \tag{88}$$

Similarly, the endowments are given by:

$$(e^1, e^2) = \{e_t^1(s_t),\, e_t^2(s_t)\}_{t=0,\, s_t \in S_t}^{\infty}. \tag{89}$$

Assuming that preferences admit a von Neumann-Morgenstern representation, the households' preferences are represented by:

$$u(c^i) = \sum_{t=0}^{\infty} \sum_{s_t \in S_t} \beta^t \pi_t(s_t) U(c_t^i(s_t)). \tag{90}$$

21.1 Arrow-Debreu Market Structure

Under the Arrow-Debreu framework, trade occurs at period 0 before any uncertainty is resolved (i.e., before s_0 is realized). Prices must be indexed by both time and event histories. Let $p_t(s_t)$ denote the price, quoted in period 0, for one unit of consumption delivered at time t if (and only if) the event history s_t occurs. With this notation, the definition of an Arrow-Debreu competitive equilibrium is analogous to the case without risk, except that the household's budget constraint involves summations over both time and event histories.

21.2 Arrow-Debreu Competitive Equilibrium

An Arrow-Debreu equilibrium consists of prices $\{\hat{p}_t(s_t)\}_{t=0,\, s_t \in S_t}^{\infty}$ and allocations $\{\hat{c}_t^i(s_t)\}_{t=0,\, s_t \in S_t}^{\infty}$ for $i = 1, 2$, such that:

(1) Given the prices $\{\hat{p}_t(s_t)\}_{t=0,\, s_t \in S_t}^{\infty}$, each agent i's allocation $\{\hat{c}_t^i(s_t)\}_{t=0,\, s_t \in S_t}^{\infty}$ solves

$$\max_{\{c_t^i(s_t)\}_{t=0,\, s_t \in S_t}^{\infty}} \sum_{t=0}^{\infty} \sum_{s_t \in S_t} \beta^t \pi_t(s_t) U(c_t^i(s_t)) \tag{3}$$

subject to

$$\sum_{t=0}^{\infty} \sum_{s_t \in S_t} \hat{p}_t(s_t) c_t^i(s_t) \leq \sum_{t=0}^{\infty} \sum_{s_t \in S_t} \hat{p}_t(s_t) e_t^i(s_t) \tag{4}$$

and

$$c_t^i(s_t) \geq 0 \quad \forall\, t,\, \forall\, s_t \in S_t. \tag{5}$$

(2) Market clearing requires that

$$\hat{c}_t^1(s_t) + \hat{c}_t^2(s_t) = e_t^1(s_t) + e_t^2(s_t) \quad \forall\, t,\ \forall\, s_t \in S_t. \tag{6}$$

It is customary to normalize the price of one commodity to 1. Note that consumption at the same date but in different event histories is treated as a distinct commodity. Moreover, although equilibrium prices reflect the probabilities of different event histories, the budget constraint does not incorporate probabilities directly.

21.3 Solving for the Equilibrium

The first-order conditions (FOCs) for consumption at period 0 and at period t in history s^t are:

$$\beta^t \pi_t(s^t) U'(c_t^i(s^t)) = \mu\, p_t(s^t)$$

and

$$\pi_0(s^0) U'(c_0^i(s^0)) = \mu\, p_0(s^0).$$

Combining these conditions yields

$$\frac{p_t(s^t)}{p_0(s^0)} = \beta^t \frac{\pi_t(s^t)}{\pi_0(s^0)} \frac{U'(c_t^i(s^t))}{U'(c_0^i(s^0))} \tag{7)(91}$$

for all t, all histories s^t, and for all agents i. This further implies that

$$\frac{U'(c_t^2(s^t))}{U'(c_t^1(s^t))} = \frac{U'(c_0^2(s^0))}{U'(c_0^1(s^0))} \tag{92}$$

for all histories s^t. For example, if a CRRA utility function is assumed, the ratio of consumption between the two agents remains constant.

We begin by considering an economy with heterogeneous agents who share the aggregate endowment. Denote the aggregate endowment by

$$e_t(s^t) = \sum_i e_t^i(s^t).$$

The resource constraint implies that each agent's consumption is a constant share of the aggregate endowment, so that for both agents

$$c_t^i(s^t) = \theta^i e_t(s^t) \tag{8)(93}$$

where θ^i represents the constant share of the aggregate endowment consumed by household i.

Using the normalization $p_0(s^0) = 1$ and equation (7), the pricing kernel is given by

$$p_t(s^t) = \beta^t \frac{\pi_t(s^t)}{\pi_0(s^0)} \frac{U'(c_t^i(s^t))}{U'(c_0^i(s^0))} \tag{9)(94}$$

Under a CRRA utility specification, this expression becomes

$$p_t(s^t) = \beta^t \frac{\pi_t(s^t)}{\pi_0(s^0)} \left(\frac{c_t^i(s^t)}{c_0^i(s^0)} \right)^{-\sigma} = \beta^t \frac{\pi_t(s^t)}{\pi_0(s^0)} \left(\frac{e_t(s^t)}{e_0(s^0)} \right)^{-\sigma} \tag{10)(95}$$

22 Lucas Asset Pricing Equation

22.1 Remarks on Consumption Pricing

The price of consumption at a given node s^t is influenced by several factors. First, it declines with time due to discounting. Second, it is higher when the node s^t is more likely to be realized. Third, it declines with the availability of resources at the node as measured by the aggregate endowment $e_t(s^t)$.

22.2 Implications for Risk Sharing

Equation (8) implies that endowment risk is perfectly shared among households. Individual household consumption is affected only by fluctuations in the aggregate endowment, while individual shocks to endowments, which do not impact the aggregate outcome (either because households are small in the aggregate or because individual endowment shocks are offset by those of others), have no effect on consumption. In this sense, the economy exhibits perfect risk sharing. Equation (10) further demonstrates that Arrow-Debreu equilibrium prices—and hence all asset prices—depend solely on the stochastic process governing the aggregate endowment, not on its distribution among households.

23 Sequential Markets Structure

23.1 Market Setup and Instruments

In the sequential markets framework, trade takes place in each period (more precisely, in each period and for every event-history pair). The model introduces one-period contingent IOUs, or Arrow securities, which are financial contracts purchased in period t that pay one unit of the consumption good in period $t+1$, but only if a particular realization $s^{t+1} = j$ occurs. Let

$$q_t(s^t; s^{t+1} = j)$$

denote the price at period t of a contract that pays out one unit of consumption in period $t+1$ if (and only if) the next period's event is $s^{t+1} = j$. These securities are alternatively known as Arrow securities, contingent claims, or one-period insurance contracts.

Let

$$a_{t+1}^i(s^t; s^{t+1})$$

denote the quantity of Arrow securities bought (or sold) at period t by agent i.

23.2 Budget Constraint in Sequential Markets

The period t, event-history s^t budget constraint for agent i is given by

$$c_t^i(s^t) + \sum_{s^{t+1}\in S} q_t(s^t, s^{t+1}) a_{t+1}^i(s^t, s^{t+1}) \leq e_t^i(s^t) + a_t^i(s^t).$$

Agents purchase Arrow securities for all contingencies that may occur in the next period. Once the state s^{t+1} is realized, only the corresponding security position becomes the asset carried forward. We assume that the initial asset position is zero, i.e., $a_0^i(s^0) = 0$ for all $s^0 \in S$.

24 Sequential Markets Competitive Equilibrium

24.1 Definition and Optimization Problem

A Sequential Markets (SM) competitive equilibrium consists of allocations

$$\{(\hat{c}_t^i(s^t), \{\hat{a}_{t+1}^i(s^t, s^{t+1})\}_{s^{t+1}\in S})_{i=1,2}\}_{t=0,\, s^t\in S^t}^{\infty}$$

and Arrow security prices

$$\{\hat{q}_t(s^t, s^{t+1})\}_{t=0,\, s^t\in S^t,\, s^{t+1}\in S}^{\infty},$$

such that the following conditions hold:

(E1) Individual Optimization: For each agent i, given the prices $\{\hat{q}_t(s^t, s^{t+1})\}$, the allocation solves

$$\max_{\{(c_t^i(s^t),\{a_{t+1}^i(s^t,s^{t+1})\}_{s^{t+1}\in S})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\sum_{s^t\in S^t} \beta^t \pi_t(s^t) u(c_t^i(s^t)) \tag{11}$$

$$\text{subject to} \quad c_t^i(s^t) + \sum_{s^{t+1}\in S} \hat{q}_t(s^t, s^{t+1}) a_{t+1}^i(s^t, s^{t+1}) \leq e_t^i(s^t) + a_t^i(s^t) \tag{12}$$

$$c_t^i(s^t) \geq 0 \quad \forall t,\, \forall s^t \in S^t \tag{13}$$

$$a_{t+1}^i(s^t, s^{t+1}) \geq -\bar{A}^i \quad \forall t,\, \forall s^t \in S^t \tag{14}$$

(E2) Market Clearing: For all periods $t \geq 0$ and every history $s^t \in S^t$, the following conditions must hold:

$$\sum_{i=1}^{2} \hat{c}_t^i(s^t) = \sum_{i=1}^{2} e_t^i(s^t) \tag{15}$$

$$\sum_{i=1}^{2} \hat{a}_{t+1}^i(s^t, s^{t+1}) = 0 \quad \forall s^{t+1} \in S \tag{16}$$

24.2 Equivalence with Arrow-Debreu Equilibrium

It can be shown that the equilibrium outcomes in the sequential markets framework are equivalent to those in the Arrow-Debreu setting. In particular, the relationship between the two sets of prices is given by

$$q_t(s^t, s^{t+1}) = \frac{p_{t+1}(s^{t+1})}{p_t(s^t)}.$$

25 Review of Markov Chains

25.1 Definition and Notation

Consider a stochastic process $\{x_t\}_{t=0}^{\infty}$ where $x_t \in X$ and the set $X = \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$ is finite. A stationary Markov chain is defined by the state space X, a transition matrix $P_{n\times n}$, and an initial probability distribution π_0 for x_0. The elements of the transition matrix, denoted by

$$P_{ij} = \Pr[x_{t+1} = \bar{x}_j \mid x_t = \bar{x}_i],$$

are independent of time.

25.2 Multi-Period Transitions

The probability of transitioning two periods ahead is given by

$$\Pr[x_{t+2} = \bar{x}_j \mid x_t = \bar{x}_i] = \sum_{k=1}^{n} P_{ik}P_{kj} \equiv [P^2]_{ij}.$$

For a given initial distribution π_0, the distribution at time t is

$$\pi_t = \pi_0 P^t,$$

and the process satisfies

$$\pi_{t+1} = \pi_t P.$$

A stationary (or invariant) distribution π satisfies

$$\pi = \pi P.$$

25.3 Intuition and History Dependence

Time is discrete and runs indefinitely. Let s_t denote the realization of a stochastic event at time t, with $s_t \in S$, where S is the finite set of possible events. Denote the history of events up to time t by

$$s^t = (s_1, s_2, \ldots, s_t),$$

and let S^t be the set of such histories. The probability of a history s^t is given by $\pi(s^t)$, with the property that $\sum_{s^t \in S^t} \pi(s^t) = 1$ for any t.

The key intuition behind Markov chains is that a first-order Markov process is one in which the current state suffices to determine the probabilities of future events. For example, the probability of rain tomorrow depends solely on whether it rained today, and not on the weather of previous days:

$$\Pr(s_{t+1} \mid s_t, s_{t-1}, \ldots, s_0) = \Pr(s_{t+1} \mid s_t) \equiv P(s'|s).$$

This property simplifies the representation of the joint distribution $\pi(s^t)$, as it can be decomposed into a product of one-period conditional probabilities.

26 Stochastic Neoclassical Growth Model (NGM)

26.1 Introduction and Environment

The stochastic neoclassical growth model (NGM) extends the deterministic version by introducing uncertainty, typically in the form of technology shocks affecting total factor productivity (TFP). This model is a cornerstone of modern business cycle theory and underpins many DSGE models used in both real business cycle and new-Keynesian frameworks.

The economy is populated by a continuum of identical households (normalized to 1 for simplicity). In each period, three goods are traded: labor services n_t, capital services k_t, and the final output good y_t, which can be allocated to consumption c_t or investment i_t. The aggregate production function is assumed to have constant returns to scale and is subject to a TFP shock:

$$F_t(k_t, 1) = z_t f(k_t),$$

where $z_t \in Z$ and Z^t represents the t-fold Cartesian product of Z.

26.2 Incorporating Uncertainty

The technology shock z_t is publicly observable and constitutes the sole source of uncertainty in the model. Denote by z^t the history of realizations up to time t, i.e.,

$$z^t = (z_t, z_{t-1}, \ldots, z_0).$$

The probability of a particular history is given by $\pi(z^t)$. Under the first-order Markov assumption, the probability of the next period's shock given the history simplifies to

$$\pi[(z_{t+1}, z^t) \mid z^t] = \pi[(z_{t+1}, z^t) \mid z_t].$$

26.3 Sequential Formulation of the Planner's Problem

The social planner's problem in the sequential formulation is written as

$$\max_{\{c_t(z^t),\, k_{t+1}(z^t)\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi(z^t) u[c_t(z^t)] \tag{96}$$

$$\text{subject to} \quad c_t(z^t) + k_{t+1}(z^t) = z_t f[k_t(z^{t-1})] + (1-\delta)k_t(z^{t-1}), \quad \forall (t, z^t), \tag{97}$$

with k_0 given.

Taking the first-order condition (FOC) with respect to k_{t+1} leads to

$$-\pi(z^t)u'[c_t(z^t)] + \sum_{z_{t+1}\in Z^{t+1}} \beta\,\pi(z^{t+1}, z^t)u'[c_{t+1}(z^{t+1}, z^t)] \\ \times\Big[z_{t+1}f'[k_{t+1}(z^t)] + (1-\delta)\Big] = 0. \tag{98}$$

Defining the conditional probability

$$\pi[(z_{t+1}, z^t)|z^t] \equiv \frac{\pi(z_{t+1}, z^t)}{\pi(z^t)},$$

the FOC can be rewritten as

$$u'[c_t(z^t)] = \sum_{z_{t+1}\in Z^{t+1}} \beta\,\pi[(z_{t+1}, z^t)|z^t]u'[c_{t+1}(z^{t+1}, z^t)] \\ \times\Big[z_{t+1}f'[k_{t+1}(z^t)] + (1-\delta)\Big] \equiv \beta\,E_{z^t}\Big[u'[c_{t+1}]R_{t+1}\Big], \tag{99}$$

where

$$R_{t+1} \equiv z_{t+1}f'[k_{t+1}(z^t)] + (1-\delta).$$

26.4 Recursive Formulation

The recursive formulation of the planner's problem is given by the Bellman equation:

$$V(k,z) = \max_{k'}\left\{u\Big[zf(k) - k' + (1-\delta)k\Big] + \beta\sum_{z'\in Z}\pi(z'|z)V(k',z')\right\}. \tag{100}$$

The solution to this problem defines the policy function $k' = g(k,z)$.

26.5 Solution and Implementation

In both the sequential and recursive formulations, the solution yields a nonlinear, stochastic difference equation—the stochastic Euler equation:

$$u'[c_t] = \beta\,E_{z_t}\Big[u'(c_{t+1})\Big[z_{t+1}f'(k_{t+1}) + 1 - \delta\Big]\Big]. \tag{101}$$

Due to its nonlinearity and stochastic nature, numerical methods or linearization techniques (around the deterministic steady state) are typically employed to solve the model.

27 Business Cycles: Introduction and Facts

27.1 Introduction to Business Cycles

Business cycles refer to fluctuations of output around its long-term growth trend. Fundamental questions in this area include understanding the origins and propagation mechanisms of these cycles. Early contributions to the theory of business cycles include the works of Lucas (1977), Kydland and Prescott (1982), and Long and Plosser (1983). The central aim is to develop models that not only offer theoretical insights but also quantitatively replicate observed facts.

One common view is that technology shocks, which affect the productivity of factors, are the primary drivers of business cycles. Although the precise definition of a technology shock can be somewhat ambiguous, it is commonly measured by the de-trended Solow residual. Besides technology shocks, alternative explanations include New Keynesian models—which introduce frictions such as price stickiness—and sunspots theories, where self-fulfilling equilibria lead to fluctuations even in the absence of fundamental changes. More recent developments incorporate informational frictions that may account for phenomena like unemployment and the role of money in the business cycle.

27.2 Empirical Facts: Volatilities and Correlations

27.2.1 Volatilities

The volatility of an economic variable x is defined as

$$\sigma_x \equiv \frac{\sqrt{\text{Var}(x)}}{\mu_x}.$$

Empirical evidence suggests the following:

- Consumption is less volatile than output, i.e., $\sigma_c < \sigma_y$.
- Consumption of durable goods exhibits higher volatility than output, $\sigma_{cd} > \sigma_y$.
- Investment is particularly volatile, with $\sigma_i \approx 3 \times \sigma_y$.
- The trade balance is more volatile than output, $\sigma_{TB} > \sigma_y$.
- Total hours worked and employment have volatilities approximately equal to that of output, $\sigma_N \approx \sigma_y$ and $\sigma_E \approx \sigma_y$ respectively.
- Hours per week are less volatile than output, $\sigma_{N_w} < \sigma_y$.
- The capital stock is much less volatile than output, $\sigma_k \ll \sigma_y$.
- The real wage is less volatile than output per hour worked, $\sigma_w \ll \sigma_{y/N}$.

27.2.2 Correlations

The following correlations have been observed empirically:

- The correlation between output per worker $\frac{y}{N}$ and output y is positive, $\rho\left(\frac{y}{N}, y\right) > 0$.
- The correlation between the real wage w and output y is approximately zero, $\rho(w, y) \approx 0$.
- The correlation between the capital stock K and output y is also approximately zero, $\rho(K, y) \approx 0$.
- The correlation between prices P and output y is negative (at least in the post-war period), $\rho(P, y) < 0$.

27.2.3 Persistence and Leads/Lags

Empirical studies indicate a high degree of persistence in output fluctuations, with the autocorrelation of output deviations approximately equal to 0.9:

$$\rho\left[(y_t - \bar{y}_t), (y_{t-1} - \bar{y}_{t-1})\right] \approx 0.9.$$

There is no clear consensus on whether consumption or investment leads output.

27.3 Methodology

In this section we describe the overall approach used in our analysis. The methodology involves the following steps.

27.4 Model Specification

1. Specify a model, including functional forms and parameters.
2. Pick parameters through calibration.
3. Solve the model numerically.
4. Simulate the model and analyze the outcome.

The simulation step involves using a random number generator to simulate a realization of the stochastic shock. This procedure produces a time series for each variable, which constitutes the researcher's "data set." Sample moments of the variables (in general, second moments) are then computed and compared to actual data.

28 Baseline Business Cycle Model

In our baseline business cycle model, we consider the central planner's problem. For example, if production is affected by a shock on total factor productivity that follows an AR(1) process, the planner's problem is given by:

$$\max_{\{c_t,n_t,l_t,K_{t+1}\}_{t=0}^{\infty}} \left\{E_0\left[\beta^t u(c_t, 1-n_t)\right]\right\} \tag{102}$$

subject to

$$y_t = f(k_t, n_t, z_t) = c_t + i_t, \tag{103}$$
$$k_{t+1} = (1-\delta)k_t + i_t, \tag{104}$$
$$\log(z_t) = \theta \log(z_{t-1}) + \epsilon_t \quad \text{(AR(1))}. \tag{105}$$

We assume that the production function f has constant returns to scale (CRS) and satisfies:

$$f_{11}, f_{22} < 0, \quad f_{11}f_{22} - f_{12}^2 = 0,$$

and that the utility function satisfies

$$u_{11}, u_{22} < 0, \quad u_{12} = 0.$$

28.1 First-Order Conditions

The first-order conditions (FOC's) for this problem are given by:

$$0 = u_1(c) - \beta E\left\{u_1(c')\left[f_1(k', n', z') + (1-\delta)\right]\right\}, \tag{1}$$
$$0 = u_2(1-n) - u_1(c) f_2(k, n, z), \tag{106}$$
$$c = f(k, n, z) + k(1-\delta) - k', \tag{107}$$
$$c' = f(k', n', z') + k'(1-\delta) - k''. \tag{108}$$

28.2 Steady State Analysis

The steady state of the model is defined by:

$$1 = \beta\left[f_1(k^*, n^*, 1) + (1-\delta)\right], \tag{2}$$
$$u_2(1-n^*) = u_1(c^*) f_2(k^*, n^*, 1), \tag{109}$$
$$y = f(k^*, n^*, 1), \tag{110}$$
$$i^* = \delta k^*, \tag{111}$$
$$c^* = y^* - i^*. \tag{112}$$

This system comprises three equations in the three unknowns: k^*, n^*, and c^*.

28.3 Linearization Around the Steady State

We now linearize the two first-order conditions around the steady state. Define the following notation:

$$u_1 \equiv u_1(c^*, 1-n^*),$$
$$u_{11} \equiv u_{11}(c^*, 1-n^*),$$
$$f_i \equiv f_i(k^*, n^*, 1), \quad i = 1, 2, 3,$$
$$f_{ij} \equiv f_{ij}(k^*, n^*, 1), \quad i, j = 1, 2, 3.$$

28.3.1 Differentiation of the FOC's

Taking the differential of the FOC's we have:

$$0 = u_{11}\,dc - \beta E\Big\{u_{11}\,dc'\,(f_1 + 1 - \delta) + u_1\,[f_{11}\,dk' + f_{12}\,dn' + f_{13}\,dz']\Big\},$$

$$\text{or}\quad 0 = u_{11}\,dc - E\Big\{u_{11}\,dc' + \beta u_1\,[f_{11}\,dk' + f_{12}\,dn' + f_{13}\,dz']\Big\},$$

$$0 = -u_{22}\,dn - f_2 u_{11}\,dc - u_1\,(f_{21}\,dk + f_{22}\,dn + f_{23}\,dz)\,.$$

28.3.2 Differentiation of the Consumption Equation

Differentiating the consumption equation yields:

$$\begin{aligned} dc &= (f_1 + 1 - \delta)\,dk + f_2\,dn + f_3\,dz - dk' \\ &= \frac{1}{\beta}\,dk + f_2\,dn + f_3\,dz - dk', \\ dc' &= (f_1 + 1 - \delta)\,dk' + f_2\,dn' + f_3\,dz' - dk'' \\ &= \frac{1}{\beta}\,dk' + f_2\,dn' + f_3\,dz' - dk''. \end{aligned}$$

28.3.3 Substitution into the Euler Equation

Substituting dc and dc' into the Euler equation and dividing by u_{11}, we obtain:

$$\begin{aligned} 0 = {} & \frac{1}{\beta}\,dk + f_2\,dn + f_3\,dz - dk' \\ & - E\Big\{\frac{1}{\beta}\,dk' + f_2\,dn' + f_3\,dz' - dk'' + \beta\frac{u_1}{u_{11}}\,[f_{11}\,dk' + f_{12}\,dn' + f_{13}\,dz']\Big\}, \end{aligned}$$

or equivalently,

$$\begin{aligned} 0 = {} & \frac{1}{\beta}\,dk + f_2\,dn + f_3\,dz - \left(1 + \frac{1}{\beta} + \beta\frac{u_1}{u_{11}}f_{11}\right)dk' \\ & - \left(f_2 + \beta\frac{u_1}{u_{11}}f_{12}\right)E\,dn' - \left(f_3 + \beta\frac{u_1}{u_{11}}f_{13}\right)E\,dz' + E\,dk''. \end{aligned}$$

Furthermore, we also have:

$$0 = -u_{22}\,dn - f_2 u_{11}\left[\frac{1}{\beta}\,dk + f_2\,dn + f_3\,dz - dk'\right]. \tag{113}$$

28.3.4 Comments on the System

Equation (3) involves dk, dk', and dk'', as well as terms in dn, dn', dz, and dz'. Additionally, the optimality condition for labor, given in (4), links dn with dk, dk', and dz. This system must be solved simultaneously.

29 Solution Methods

In this section we describe the procedure for solving the model.

29.1 Non-Stochastic Model and Steady State

Consider first the non-stochastic version of the model. By obtaining the Euler equation and substituting the resource constraint, the solution takes the form

$$v(K_t, K_{t+1}, K_{t+2}) = 0, \quad t = 0, 1, \dots \quad \text{with } K_0 \text{ given.}$$

A solution for this model is a function $K_{t+1} = g(K_t)$ that satisfies the above equation. Once the steady state is determined, the model is linearized around it.

29.2 Difference Equation Formulation

The linearized model yields an equation of the form

$$\tilde{k}_{t+2} - \phi\tilde{k}_{t+1} + \frac{1}{\beta}\tilde{k}_t = 0,$$

where the variables are expressed in terms of deviations from the steady state. Assume that $\tilde{k}_0 \neq 0$, $0 < \beta < 1$, and $\phi > 1 + \frac{1}{\beta}$. This sequence of difference equations defines the solution $\{k_t\}_{t=0}^{\infty}$.

29.2.1 Dimensionality of the Solution

The sequence is completely determined by the value of k_1, hence the set of solutions is one-dimensional, parameterized by k_1. An alternative characterization is obtained by assuming that the policy function is linear, i.e.,

$$k_{t+1} = \lambda k_t.$$

Then, the characteristic equation becomes

$$f(\lambda) = \lambda^2 - \phi\lambda + \frac{1}{\beta}.$$

29.2.2 Explosive and Non-Explosive Solutions

There exist two solutions in which k_t solves a first order difference equation:

$$\tilde{k}_t = \tilde{k}_0\lambda_1^t,$$

$$\tilde{k}_t = \tilde{k}_0\lambda_2^t.$$

The solutions are non-explosive if

$$\lim_{t\to\infty} \tilde{k}_t = 0.$$

There are three cases to consider:

- Case 1: $\lambda_1 > 1$, $\lambda_2 < 1$.
- Case 2: $\lambda_1 > 1$, $\lambda_2 > 1$.
- Case 3: $\lambda_1 < 1$, $\lambda_2 < 1$.

29.3 General Case: Rational Expectations Models

Rational expectations models are generally challenging to solve, even numerically. Many such models can be expressed as systems of linear stochastic first order difference equations. Linearity may be imposed by approximation, and a first order autoregressive structure can be achieved by appropriately renaming lagged variables of higher order. Several methods have been developed, including those by Blanchard and Khan (1980), Uhlig (1999), Christiano (2002), Klein (2000), and Sims (2002).

30 Method of Undetermined Coefficients

The following sections describe the method of undetermined coefficients as presented in Christiano (2002).

30.1 Model Setup

Consider the following model:

$$\begin{aligned} \alpha_0 E_t x_{t+1} + \alpha_1 x_t + \alpha_2 x_{t-1} + \beta z_t = 0, \quad t \geq 0, \quad x_{-1} \text{ given}, \\ z_t = R z_{t-1} + \epsilon_t, \\ \epsilon_t \sim N(0, \Sigma) \quad \text{i.i.d.} \end{aligned}$$

Here:

- x_t is an $n \times 1$ vector of endogenous variables determined at time t.
- z_t is a $k \times 1$ vector of exogenous stationary shocks.
- The matrices α_0, α_1, and α_2 are $n \times n$, β is $n \times k$, and R is $k \times k$.
- $E_t(\Delta)$ denotes the mathematical expectation conditional on the information available at time t, which includes current and past values of x and z.

30.2 Linear Feedback Solution

A solution to the model is proposed as a feedback rule relating the current endogenous vector x_t to its past value x_{t-1} and the current exogenous shock z_t. Since the model is linear, we postulate a linear solution:

$$x_t = A x_{t-1} + B z_t, \tag{7}$$

with the restriction that for stationary equilibria, all eigenvalues of A must have absolute value less than 1. The objective is to determine matrices A and B such that, given x_{t-1}, equation (7) is consistent with (5) and the stationarity of x_t.

30.2.1 Decomposition of the Problem

Substituting (7) into (5) and using the fact that $E_t(z_{t+1}) = R z_t$, we obtain:

$$(\alpha_0 A^2 + \alpha_1 A + \alpha_2) x_{t-1} + \Big[(\alpha_0 A + \alpha_1) B + \alpha_0 B R + \beta\Big] z_t = 0. \tag{8}$$

Since this must hold for any realization of z_t, it is natural to solve the system in two steps:

1. First, shut down the exogenous shock ($z_t = 0$) and solve for the matrix A, i.e.,

   $$(\alpha_0 A^2 + \alpha_1 A + \alpha_2) x_{t-1} = 0. \tag{9}$$

   This implies

   $$\alpha_0 A^2 + \alpha_1 A + \alpha_2 = 0. \tag{10}$$

2. Given A, solve for B from:

   $$(\alpha_0 A + \alpha_1) B + \alpha_0 B R + \beta = 0. \tag{11}$$

30.2.2 Solution for A

Equation (10) is quadratic in A and may have multiple solutions. The number of solutions that satisfy the stationarity restriction (all eigenvalues less than 1 in absolute value) determines the existence and uniqueness of the solution.

Non-Stochastic Case: If $z_t = 0$ for all t, the model reduces to:

$$\Gamma_0 Y_{t+1} + \Gamma_1 Y_t = 0, \quad t \geq 0, \tag{12}$$

where

$$Y_t = \begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix}, \quad \Gamma_0 \equiv \begin{bmatrix} \alpha_0 & 0_{n\times n} \\ 0_{n\times n} & I_n \end{bmatrix}, \quad \Gamma_1 \equiv \begin{bmatrix} \alpha_1 & \alpha_2 \\ -I_n & 0_{n\times n} \end{bmatrix}. \tag{114}$$

Assuming that Γ_0 is invertible (i.e., α_0 is invertible), we have:

$$Y_{t+1} = -\Gamma_0^{-1}\Gamma_1 Y_t. \tag{13}$$

Eigenvalue Decomposition: Assume that $-\Gamma_0^{-1}\Gamma_1$ has $2n$ linearly independent eigenvectors. Then we can write:

$$-\Gamma_0^{-1}\Gamma_1 = P\Lambda P^{-1}, \tag{14}$$

where Λ is a diagonal matrix containing the eigenvalues and P is the corresponding eigenvector matrix. Let $\bar{n}$ denote the number of stable eigenvalues (i.e., those with absolute value less than 1). Define Λ_s as the diagonal matrix with the stable eigenvalues and Λ_e with the explosive eigenvalues. Then Λ can be partitioned as:

$$\Lambda = \begin{bmatrix} \Lambda_s^{\bar{n}\times\bar{n}} & 0^{\bar{n}\times(2n-\bar{n})} \\ 0^{(2n-\bar{n})\times\bar{n}} & \Lambda_e^{(2n-\bar{n})\times(2n-\bar{n})} \end{bmatrix}. \tag{15}$$

Define $W_t = P^{-1}Y_t$ so that

$$W_t = \Lambda W_{t-1}. \tag{16}$$

Iterating this yields:

$$W_t = \begin{bmatrix} \Lambda_s^t & 0 \\ 0 & \Lambda_e^t \end{bmatrix} W_0. \tag{17}$$

Since W_t is a linear function of x_t and x_{t-1}, stationarity of x_t requires that W_t is also stationary. The first $\bar{n}$ components of W_t will tend to zero as $t \to \infty$; however, the lower $2n - \bar{n}$ components may be explosive. To ensure stationarity, the corresponding elements in W_0 must be set equal to zero.

Assuming $\bar{n} = n$, we write:

$$\begin{bmatrix} W_{1,0} \\ W_{2,0} \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ x_{-1} \end{bmatrix}, \tag{115}$$

where $W_{1,0}$ consists of the first n elements and $W_{2,0}$ consists of the last n elements. For stationarity, we impose:

$$P_{21}x_0 + P_{22}x_{-1} = 0, \tag{18}$$

and, provided that P_{21} is invertible,

$$x_0 = -(P_{21})^{-1}P_{22}x_{-1}. \tag{19}$$

Thus, we obtain the matrix A as:

$$A = -(P_{21})^{-1}P_{22}. \tag{20}$$

30.2.3 Solution for B

Once A is determined, equation (11) is linear in B. The solution is given by:

$$\mathrm{vec}(B) = -\left[I_k \otimes (\alpha_0 A + \alpha_1) + R' \otimes \alpha_0\right]^{-1} \mathrm{vec}(\beta), \tag{21}$$

where the vec operator vectorizes the matrix B (stacking its columns into a vector) and $\otimes$ denotes the Kronecker product.

30.3 Existence and Uniqueness Considerations

In the derivation above, we assumed that $\bar{n} = n$ (i.e., the number of stable eigenvalues equals the number of elements in x_t). The following cases arise:

1. If $\bar{n} < n$: One must zero out $2n - \bar{n} > n$ elements with only n free variables, so a stationary equilibrium does not exist.
2. If $\bar{n} > n$: One must zero out $2n - \bar{n} < n$ elements with n free variables, implying that there exist many values of x_0 consistent with a stationary equilibrium; that is, equilibrium is not unique.

30.4 Linearized Solution of the Model

The linearized solution of the model can be written in the form:

$$E_t X_{t+1} = M X_t, \quad t \geq 0, \tag{22}$$

where

$$X_{t+1} = \begin{bmatrix} x_{1,t+1}^{n\times 1} \\ x_{2,t+1}^{m\times 1} \end{bmatrix}. \tag{116}$$

Here, n represents the number of "jump" or "forward-looking" variables and m represents the number of state variables. For example, in the one-sector growth model with TFP shocks, $n = 1$ and $m = 2$.

The matrix M contains the parameters of the model. However, note that this only describes how the forward-looking variables evolve; the initial condition must still be imposed to ensure non-explosiveness.

30.4.1 Eigenvalue and Eigenvector Characterization

By definition, the eigenvalue problem for M is:

$$Mv = \lambda v, \tag{23}$$

or equivalently,

$$(M - \lambda I)v = 0. \tag{24}$$

There will be $n + m$ eigenvalues and corresponding eigenvectors, which can be organized as follows:

$$M \begin{bmatrix} v_{1,1} & \cdots & v_{n+m,1} \\ v_{1,2} & \cdots & v_{n+m,2} \\ \vdots & \ddots & \vdots \\ v_{1,n+m} & \cdots & v_{n+m,n+m} \end{bmatrix} = \begin{bmatrix} v_{1,1} & \cdots & v_{n+m,1} \\ v_{1,2} & \cdots & v_{n+m,2} \\ \vdots & \ddots & \vdots \\ v_{1,n+m} & \cdots & v_{n+m,n+m} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{n+m} \end{bmatrix}. \tag{25}$$

This completes the overview of the method of undetermined coefficients.

31 Theoretical Framework and Solution Method

31.1 Matrix Representation and Diagonalization

We start by redefining the matrices:

$$M\Gamma\Lambda = \Gamma\Lambda \tag{117}$$

thus,

$$M = \Gamma\Lambda\Gamma^{-1}. \tag{118}$$

Let Q denote the number of stable eigenvalues (absolute value less than 1). For convenience, let Λ_1 be a $Q \times Q$ diagonal matrix with the stable eigenvalues along its diagonal, and let Λ_2 be a $B \times B$ diagonal matrix with the explosive eigenvalues. Hence,

$$\Lambda = \begin{bmatrix} \Lambda_1^{Q\times Q} & 0 \\ 0 & \Lambda_2^{B\times B} \end{bmatrix}. \tag{119}$$

31.2 Transformation of the System

We can now write the system as

$$E_t X_{t+1} = \Gamma\Lambda\Gamma^{-1} X_t, \tag{120}$$

$$E_t \Gamma^{-1} X_{t+1} = \Lambda\Gamma^{-1} X_t. \tag{121}$$

Define an auxiliary vector by

$$Z_t = \Gamma^{-1} X_t. \tag{122}$$

The system is then rewritten as

$$E_t Z_{t+1} = \Lambda Z_t, \tag{123}$$

$$E_t \begin{bmatrix} Z_{1,t+1} \\ Z_{2,t+1} \end{bmatrix} = \begin{bmatrix} \Lambda_1^{Q\times Q} & 0 \\ 0 & \Lambda_2^{B\times B} \end{bmatrix} \begin{bmatrix} Z_{1,t} \\ Z_{2,t} \end{bmatrix}. \tag{124}$$

This represents a VAR(1) system with a diagonal coefficient matrix, so that

$$E_t Z_{1,t+T} = \Lambda_1^T Z_{1,t}, \tag{125}$$

$$E_t Z_{2,t+T} = \Lambda_2^T Z_{2,t}. \tag{126}$$

Notice that:

- $\Lambda_1^T \to 0$ as $T \to \infty$.
- $E_t Z_{2,t+T} \to \infty$ as $T \to \infty$ unless $Z_{2,t} = 0$.
- To satisfy the transversality condition and/or the feasibility constraint, we must have $Z_{2,t} = 0$.

31.3 Decomposition of Γ^{-1} and Policy Function

Write Γ^{-1} as

$$\Gamma^{-1} = \begin{bmatrix} G_{1,1}^{Q\times n} & G_{1,2}^{Q\times m} \\ G_{2,1}^{B\times n} & G_{2,2}^{B\times m} \end{bmatrix}. \tag{127}$$

Then, given the definition of Z_t, we have:

$$Z_{1,t}^{Q\times 1} = G_{1,1}\, x_{1,t}^{n\times 1} + G_{1,2}\, x_{2,t}^{m\times 1}, \tag{128}$$

$$Z_{2,t}^{B\times 1} = G_{2,1}\, x_{1,t}^{n\times 1} + G_{2,2}\, x_{2,t}^{m\times 1}. \tag{129}$$

To ensure stability, we need $Z_{2,t} = 0$, so

$$0^{B\times 1} = G_{2,1}\, x_{1,t}^{n\times 1} + G_{2,2}\, x_{2,t}^{m\times 1}, \tag{130}$$

which implies

$$G_{2,1}\, x_{1,t} = -G_{2,2}\, x_{2,t}. \tag{131}$$

If $G_{2,1}$ is square, the linearized policy function is obtained as:

$$x_{1,t} = -G_{2,1}^{-1} G_{2,2}\, x_{2,t}. \tag{132}$$

An important requirement here is that $B = n$, ensuring that the number of unstable eigenvalues (greater than one) matches the number of jump variables. If $B < n$, multiple stable (stationary) solutions arise, and if $B > n$, no solution exists.

32 Application: The Neoclassical Growth Model

32.1 The Deterministic Model Setup

Consider the non-stochastic neoclassical growth model with CRRA preferences, Cobb-Douglas production, and the level of technology normalized to unity. The equilibrium conditions are given by:

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} \left(\alpha k_{t+1}^{\alpha-1} + (1-\delta) \right), \tag{133}$$

$$k_{t+1} = k_t^{\alpha} - c_t + (1-\delta) k_t. \tag{134}$$

Log-linearizing around the steady state yields:

$$-\sigma \tilde{c}_t = -\sigma \tilde{c}_{t+1} + \beta(\alpha-1) R^* \tilde{k}_{t+1}, \tag{135}$$

$$\tilde{k}_{t+1} = \frac{1}{\beta} \tilde{k}_t - \frac{c^*}{k^*} \tilde{c}_t, \tag{136}$$

where $(\cdot)^*$ indicates the steady state value of the variable and $R^* = \alpha k^{*\alpha-1}$.

32.2 Matrix Representation and Stability Condition

The system in matrix form is:

$$\begin{bmatrix} \tilde{c}_{t+1} \\ \tilde{k}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 - \frac{c^*}{k^*} \frac{\beta(\alpha-1)R^*}{\sigma} & \frac{(\alpha-1)R^*}{\sigma} \\ -\frac{c^*}{k^*} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} \tilde{c}_t \\ \tilde{k}_t \end{bmatrix}. \tag{137}$$

Let $\sigma = 1$ (log utility), $\beta = 0.95$, $\delta = 0.1$, and $\alpha = 0.33$. With these values, after computing Γ, P^{-1}, and Λ (using Matlab or another software), imposing the condition for stability yields:

$$\tilde{c}_t = 0.5557\, \tilde{k}_t, \tag{138}$$

and consequently,

$$c_t = 0.4443\, c_t^* + 0.5557\, \frac{c^*}{k^*} k_t. \tag{139}$$

33 Real Business Cycle Models

33.1 Calibration Framework (Following Cooley and Prescott, 1995)

33.1.1 Model Calibration

We illustrate the calibration of a business cycle model using an example. The stochastic shock to total factor productivity is assumed to follow an AR(1) process; the parameters $\tilde{\rho}$ and $\tilde{\sigma}^2$ are computed from de-trended (HP-filtered) data. Preferences are assumed to be logarithmic.

33.1.2 Production Technology and Model Setup

Production is described by:

$$Y_t = \exp(z_t) K_t^{\alpha} (P_t n_t A_t)^{(1-\alpha)},$$

with:

- **Technological Change:** $A_{t+1} = (1+\gamma)A_t = (1+\gamma)^{t+1} A_0$ with $A_0 = 1$.
- **Population Growth:** $P_{t+1} = (1+\eta)P_t = (1+\eta)^{t+1} P_0$ with $P_0 = 1$.
- **Technology Shock:** z_t follows an AR(1) process, $z_{t+1} = \rho z_t + \epsilon_{t+1}$ with $\epsilon_t \sim N(0, \sigma_\epsilon^2)$.

33.1.3 Planner's Problem

The planner's problem is formulated as:

$$\max_{\{c_t, l_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t), \tag{140}$$

subject to the resource constraint:

$$C_t + X_t = \exp(z_t) K_t^{\alpha} (n_t P_t (1+\gamma)^t)^{(1-\alpha)}.$$

Dividing the resource constraint by the population size P_t gives:

$$\frac{C_t}{P_t} + \frac{X_t}{P_t} = (1+\gamma)^{t(1-\alpha)} \exp(z_t) \left(\frac{K_t}{P_t}\right)^{\alpha} n_t^{(1-\alpha)},$$

or, equivalently,

$$c_t + x_t = (1+\gamma)^{t(1-\alpha)} \exp(z_t) k_t^{\alpha} n_t^{(1-\alpha)}.$$

33.1.4 Labor and Capital Dynamics

With individual time endowment being limited,

$$l_t + n_t = 1. \tag{141}$$

The capital accumulation equation is standard:

$$K_{t+1} = (1 - \delta)K_t + X_t,$$

and dividing by P_t yields:

$$\frac{K_{t+1}}{P_t} = (1 - \delta)\frac{K_t}{P_t} + \frac{X_t}{P_t},$$
$$(1 + \eta)k_{t+1} = (1 - \delta)k_t + x_t.$$

33.2 Transforming the Model to Stationarity

To solve the model, we transform the growth model into a stationary one. Since all variables grow at rate γ, we define de-trended variables as follows:

$$\tilde{c}_t = \frac{c_t}{(1 + \gamma)^t}, \quad \tilde{x}_t = \frac{x_t}{(1 + \gamma)^t}, \quad \tilde{k}_t = \frac{k_t}{(1 + \gamma)^t}.$$

Note that by definition,

$$\log(c_t) = \log(\tilde{c}_t) + \log((1 + \gamma)^{-t}),$$

and the second term is irrelevant for optimization.

Further, transform the budget constraint by dividing it by $(1 + \gamma)^{t+1}$:

$$(1 + \gamma)(1 + \eta)\tilde{k}_{t+1} = (1 - \delta)\tilde{k}_t + \tilde{x}_t. \tag{142}$$

Also, note that $P_t = (1 + \eta)^t$.

Thus, the transformed problem becomes:

$$\max_{\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t (1 + \eta)^t \left[\log(\tilde{c}_t) + \frac{\theta}{1 - \theta}\log(l_t)\right],$$

subject to:

$$\tilde{c}_t + (1 + \gamma)(1 + \eta)\tilde{k}_{t+1} = \exp(z_t)\tilde{k}_t^{\alpha}(1 - l_t)^{(1-\alpha)} + (1 - \delta)\tilde{k}_t.$$

34 Empirical Implementation: Taking the Model to Data

34.1 Calibration Methodology

The central methodological issue is how to choose the parameters in the utility and production functions. Calibration, as advocated by Kydland and Prescott (1982), involves picking parameter values from sources independent of the phenomenon under study:

1. Household data on consumption, hours worked, and other microeconomic evidence for individual preference parameters.

2. Long-run trend data for factor shares in production (notably α in the Cobb-Douglas production function).

The idea is to match moments calculated from the model to those computed from data, analogous to the method-of-moments.

34.2 National Accounts and Parameter Mapping

34.2.1 Capital Share Parameter α

Since we have constant returns to scale (CRS) production and competitive markets,

$$Y = rK + wL,$$

it follows that $\alpha = \frac{rK}{Y}$. Empirical calibration typically suggests $\alpha \approx \frac{1}{3}$.

34.2.2 Definitions from National Income and Product Accounts (NIPAs)

Gross Domestic Product (GDP): The market value of goods and services produced by labor and property located in the United States (including residents and non-residents).

Gross National Product (GNP): The market value of goods and services produced by labor and property supplied by U.S. residents.

National Income (NI): Includes all net incomes earned in production (net of CFC). It is the sum of compensation of employees, proprietors' income with inventory valuation adjustment (IVA) and capital consumption adjustment (CCAdj), rental income of persons with CCAdj, corporate profits with IVA and CCAdj, net interest and miscellaneous payments, taxes on production and imports, business current transfer payments, and the current surplus of government enterprises, less subsidies.

Net National Product (NNP): The net market value of goods and services attributable to the labor and property supplied by U.S. residents; equal to GNP less CFC. In NIPA calculations, CFC relates only to fixed capital located in the United States. Investment in capital is measured by private fixed investment and government gross investment.

34.2.3 Capital Income and Its Components

According to Cooley and Prescott (1995),

$$\alpha = \frac{\text{Capital Income}}{\text{Total Income}},$$

where Capital Income is defined as private capital income. More specifically, let

$$I_K = UI + AI + DEP, \tag{143}$$

with:

- **UI**: Unambiguous Component (rental income, corporate profits, net interest).
- **AI**: $\alpha\times$ Ambiguous Component (proprietor's income, $(NNP - NI)$).
- **DEP**: Depreciation.

Define $\alpha_M = \frac{I_K}{GNP}$. Then,

$$AI = \alpha_M \left[PI + (NNP - NI)\right]. \tag{144}$$

Thus, from the above,

$$\alpha_M = \frac{UI + \alpha_M \left[PI + (NNP - NI)\right] + DEP}{GNP}. \tag{145}$$

Solving for α_M gives:

$$\alpha_M = \frac{UI + DEP}{GNP - (PI + NNP - NI)}. \tag{146}$$

Furthermore, the gross interest rate for the economy is given by:

$$r = \frac{I_K}{K}, \tag{147}$$

and the net interest rate by:

$$i = \frac{I_K - DEP}{K}. \tag{148}$$

34.3 Additional Calibration Parameters

γ: The average long-run growth rate, typically around 2%, computed from GDP data.

η: The average population growth rate, approximately 1%.

δ: The depreciation rate, determined from the steady state of the capital accumulation equation. From

$$(1+\gamma)(1+\eta)\tilde{k}^* = (1-\delta)\tilde{k}^* + \tilde{x}^*,$$

dividing both sides by $\tilde{k}^*$ yields:

$$(1+\gamma)(1+\eta) = (1-\delta) + \frac{\tilde{x}^*}{\tilde{k}^*} = (1-\delta) + \frac{X^*}{K^*}.$$

With data indicating $\frac{X^{US}}{K^{US}} = 0.076$, it follows that $\delta = 0.0458$.

β and θ: These parameters are identified using the first-order conditions (FOC) of the planner's problem. In the steady state (balanced growth path, where $\tilde{c}_t = \tilde{c}_{t+1}$ and $z_t = 0$), the FOC with respect to next period's capital is:

$$(1+\gamma) = \beta\left[\alpha\tilde{k}_t^{\alpha-1}(1-l_t)^{1-\alpha} + 1 - \delta\right].$$

Given that $\alpha\tilde{k}_t^{\alpha-1}(1-l_t)^{1-\alpha} = \alpha\frac{\tilde{y}_t}{\tilde{k}_t} = \alpha\frac{Y_t}{K_t}$ and using data (with $\frac{Y_t^{US}}{K_t^{US}} \approx 0.3012$), and knowing α and γ, we obtain $\beta = 0.949$.

To determine θ, use the FOC with respect to $\tilde{c}_t$ and l_t:

$$\begin{aligned} \tilde{c}_t: &\quad \frac{1}{\tilde{c}_t} = \lambda_t, \\ l_t: &\quad \frac{\theta}{1-\theta}\frac{1}{l_t} = \lambda_t(1-\alpha)\exp(z_t)\tilde{k}_t^{\alpha}(1-l_t)^{-\alpha}. \end{aligned}$$

Thus, the Euler equation is:

$$\frac{\theta}{1-\theta}\frac{1-l_t}{l_t} = (1-\alpha)\frac{\tilde{y}_t}{\tilde{c}_t} = (1-\alpha)\frac{Y_t}{C_t}.$$

Assuming a daily time endowment of 24 hours, with 8 hours allocated to sleep, 8 to work, and 8 to leisure, we approximate $l_t \approx \frac{2}{3}$. Solving for θ yields $\theta = 0.61612$.

AR(1) Process Parameters: In the spirit of Kydland and Prescott (1982) and Long and Plosser (1983), z_t represents an aggregate technology shock affecting production. While there may be other shocks (e.g., preference or government-related), the calibration of z_t is inspired by Solow's growth accounting, where two inputs (capital and labor) and a total productivity shock are considered.

34.4 Calibration Equations

We have that

$$y_t = \exp(z_t) k_t^{\alpha} (n_t)^{(1-\alpha)}$$

By taking logs,

$$\log(y_t) = z_t + \alpha \log(k_t) + (1-\alpha)\log(n_t)$$

Therefore,

$$\log(z_{t+1}) - \log(z_t) = \log(y_{t+1}) - \log(y_t) + \alpha\Big[\log(k_{t+1}) - \log(k_t)\Big] + (1-\alpha)\Big[\log(n_{t+1}) - \log(n_t)\Big]$$

We can observe capital and labor in the data and since we already know α, we can construct time series for z_t and use it to estimate ρ and σ^2.

34.5 Critiques of the Calibration Approach

Some Critiques.

z may not be technology, but just poor measurement (Jorgenson-Griliches argument).

z exhibits a high variation - then what are these shocks? It should be possible to identify them. Furthermore, what is the meaning of a "negative" technological shock? Can technology somehow worsen from one period to the other?

The story of stochastic productivity shocks may be acceptable on an industry, or firm level. But the notion of aggregate technological shocks seems more dubious. An aggregation argument of individual, independent shocks cannot work either, since by the law of large numbers this should lead to no variation at all. Some kind of aggregate component is needed (correlation of individual shocks is tantamount to an aggregate effect).

34.6 Measurement Issues

Some Measurement Issues.

Unmeasured output (home production, quality, etc).

Capital is not always fully utilized (less utilized in recessions) so z could be picking that.

Human capital is not observable so it is not correctly measured. There is little information on the phenomenon known as "labor hoarding": personnel that is kept at their posts doing unproductive tasks.

35 Consumption and Asset Pricing

35.1 Introduction

We analyze the theory of consumption and asset pricing in dynamic representative agent models. In these models, consumption and the price of assets are intimately related. We start by studying consumption theory under certainty and uncertainty. Review the asset pricing formula derived by Lucas (1978). We study the application of the Lucas model done by Mehra and Prescott (1985), also known as the "equity premium puzzle".

35.2 Consumption Under Certainty

35.2.1 Theoretical Framework

The main feature of the data that consumption theory aims to explain is that aggregate consumption is smooth, relative to aggregate income. Why is consumption smoother than income? This leads to a model of consumption smoothing.

Consider a consumer with initial assets A_0 and preferences

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \tag{149}$$

where $0 < \beta < 1$, c_t is consumption, and $u(\cdot)$ is increasing, strictly concave, and twice differentiable.

The consumer's budget constraint reads

$$A_{t+1} = (1+r)(A_t - c_t + w_t) \tag{150}$$

for $t = 0, 1, 2, \ldots$, where r is the one-period interest rate (assumed constant over time) and w_t is income in period t, which is exogenous. We also assume the no-Ponzi-scheme condition

$$\lim_{t\to\infty} \frac{A_t}{(1+r)^t} = 0. \tag{151}$$

From (2) and (3), we obtain the intertemporal budget constraint:

$$\sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} = A_0 + \sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t}. \tag{152}$$

The representative consumer chooses sequences $\{c_t, A_{t+1}\}_{t=0}^{\infty}$ to maximize (1) subject to (2) and (3).

The Bellman equation for this problem is

$$v(A_t) = \max_{A_{t+1}} \Big\{ u\Big(w_t + A_t - \frac{A_{t+1}}{1+r}\Big) + \beta\, v(A_{t+1}) \Big\}. \tag{153}$$

It is easy to obtain the Euler equation:

$$\frac{u'(c_t)}{\beta\, u'(c_{t+1})} = 1 + r. \tag{154}$$

35.2.2 Example 1

Suppose $1 + r = \frac{1}{\beta}$. Then,

$$\frac{u'(c_t)}{u'(c_{t+1})} = 1, \tag{155}$$

which gives

$$c_t = c_{t+1} = c \quad \forall\, t. \tag{156}$$

Then from (4) we get that

$$c = \left(\frac{r}{1+r}\right)\left(A_0 + \sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t}\right). \tag{157}$$

35.2.3 Example 2

Suppose $u(c) = \frac{c^{1-\alpha}-1}{1-\alpha}$, with $\alpha > 0$. From the Euler equation we obtain

$$\frac{c_{t+1}}{c_t} = [\beta(1+r)]^{\frac{1}{\alpha}}, \tag{158}$$

so that consumption grows at a constant rate for all t. Thus,

$$c_t = c_0[\beta(1+r)]^{\frac{t}{\alpha}}. \tag{159}$$

Solving for c_0 using (4),

$$c_0 = \left[1 - \beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right]\left(A_0 + \sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t}\right). \tag{160}$$

35.2.4 Remarks on Consumption Smoothing

Consumption in each period is a constant fraction of discounted lifetime wealth or "permanent income". The sequence $\{w_t\}_{t=0}^{\infty}$ could be highly variable, but the consumer is able to smooth consumption perfectly by borrowing and lending in a perfect capital market. Equation (9) implies that the response of consumption to an increase in permanent income is very small. This is the permanent income hypothesis (PIH): because consumers smooth consumption over time, the impact on consumption of a temporary increase in income is very small.

35.3 Consumption Under Uncertainty

35.3.1 Model Setup

We study Hall (1978)'s model, which is a formalization of the permanent income hypothesis developed by Friedman. Consider a consumer with initial assets A_0 and preferences

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{161}$$

where $u(\cdot)$ retains the properties stated previously, and subject to the same budget constraint given by (2). The consumer's income, w_t, is a random variable which becomes known at the beginning of period t.

35.3.2 The Bellman Equation and Stochastic Euler Equation

The Bellman equation now reads

$$v(A_t, w_t) = \max_{A_{t+1}} \Big\{u\Big(w_t + A_t - \frac{A_{t+1}}{1+r}\Big) + \beta\, E_t\, v(A_{t+1}, w_{t+1})\Big\}. \tag{162}$$

It is easy to obtain the Euler equation:

$$E_t u'(c_{t+1}) = \frac{1}{\beta(1+r)} u'(c_t). \tag{163}$$

35.3.3 Remarks on Uncertainty

This stochastic Euler equation captures the implications of the PIH under uncertainty. It states that $u'(c_t)$ is a martingale with drift. For example, suppose that we have quadratic utility, i.e.,

$$u(c_t) = -\frac{1}{2}(\bar{c} - c_t)^2,$$

where $\bar{c} > 0$ is a constant. Then,

$$E_t c_{t+1} = \left[\frac{\beta(1+r) - 1}{\beta(1+r)}\right] c_t. \tag{164}$$

That is, consumption is smooth in the sense that the only information required to predict future consumption is current consumption.

There is a vast literature comparing these predictions with the data. Empirically, consumption appears much more volatile and responsive to changes in current income than the theory predicts. Several explanations have been proposed:

- Sometimes aggregate consumption data is used, and the ability of consumers to smooth consumption is determined by the investment technology. For instance, in real business cycle models, asset prices move so that all production is consumed in each period. Interest rates are not exogenous as in the Hall (1978) model, and they fit consumption data relatively well.
- Another explanation is that capital markets are not as perfect as assumed. In practice, the interest rates at which consumers can borrow are typically much higher than the rates at which they can lend, or in some cases, borrowing is not available at all (financial frictions). This limits the ability of consumers to smooth consumption, making consumption more volatile or sensitive to income changes.

35.4 Lucas (1978)'s Asset Pricing Model

35.4.1 Model Setup and Equilibrium

In Lucas (1978)'s asset pricing model, consumption is taken as exogenous while asset prices are determined endogenously. This model is sometimes referred to as the ICAPM or the consumption-based capital asset pricing model. Lucas considers an endowment economy inhabited by a representative agent with preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{165}$$

where $0 < \beta < 1$, and output is produced by n productive units (trees), where y_{it} is the random output (fruit) produced by productive unit i in period t. In equilibrium,

$$c_t = \sum_{i=1}^{n} y_{it}. \tag{166}$$

35.4.2 Competitive Equilibrium and the Budget Constraint

We consider a stock market economy in which the representative consumer is endowed with 1 share of each productive unit at time 0 and the stock of shares remains constant over time. Each period, the output on each productive unit (the dividend) is distributed to shareholders in proportion to their share holdings, and then shares are traded on competitive markets. Let

p_{it} denote the price of a share in productive unit i (in terms of the consumption good), and z_{it} denote the number of shares held at the beginning of period t. The budget constraint then reads

$$\sum_{i=1}^{n} p_{it} z_{i,t+1} + c_t = \sum_{i=1}^{n} z_{it}(p_{it} + y_{it}) \tag{167}$$

for $t = 0, 1, \ldots$.

35.4.3 Dynamic Programming Formulation

Denote by p_t, z_t, and y_t the vectors of prices, share holdings, and outputs, respectively (e.g., $y_t = (y_{1t}, y_{2t}, \ldots, y_{nt})$). The Bellman equation for the representative consumer becomes

$$v(z_t, y_t) = \max_{c_t, z_{t+1}} \Big\{ u(c_t) + \beta\, E_t\, v(z_{t+1}, y_{t+1}) \Big\}, \tag{168}$$

subject to (3). Substituting the budget constraint,

$$v(z_t, y_t) = \max_{c_t, z_{t+1}} \Big\{ u\Big(\sum_{i=1}^{n} z_{it}(p_{it} + y_{it}) - p_{it} z_{i,t+1} \Big) + \beta\, E_t\, v(z_{t+1}, y_{t+1}) \Big\}. \tag{169}$$

Taking the first-order conditions (FOC) with respect to $z_{i,t+1}$ for each $i = 1, \ldots, n$, we obtain

$$-p_{it}\, u'\Big(\sum_{i=1}^{n} z_{it}(p_{it} + y_{it}) - p_{it} z_{i,t+1} \Big) + \beta\, E_t \frac{\partial v}{\partial z_{i,t+1}} = 0. \tag{170}$$

The envelope conditions are given by

$$\frac{\partial v}{\partial z_{i,t}} = (p_{it} + y_{it})\, u'\Big(\sum_{i=1}^{n} z_{it}(p_{it} + y_{it}) - p_{it} z_{i,t+1} \Big). \tag{171}$$

Substituting the envelope conditions into (6) and using (2), we obtain

$$-p_{it}\, u'\Big(\sum_{i=1}^{n} y_{it} \Big) + \beta\, E_t \Big[(p_{i,t+1} + y_{i,t+1})\, u'\Big(\sum_{i=1}^{n} y_{i,t+1} \Big) \Big] = 0, \tag{172}$$

or equivalently,

$$p_{it} = E_t \Big[\Big(p_{i,t+1} + y_{i,t+1} \Big) \frac{\beta\, u'(c_{t+1})}{u'(c_t)} \Big]. \tag{173}$$

35.4.4 The Pricing Kernel and Asset Returns

The current price of a share is thus equal to the expectation of the product of the future payoff on that share with the intertemporal marginal rate of substitution. Define the gross rate of return of share i between period t and $t+1$ as

$$\pi_{it} = \frac{p_{i,t+1} + y_{i,t+1}}{p_{it}}, \tag{174}$$

and let

$$m_t = \frac{\beta\, u'(c_{t+1})}{u'(c_t)} \tag{175}$$

denote the stochastic discount factor or pricing kernel. Then,

$$E_t(\pi_{it}\, m_t) = 1. \tag{176}$$

By repeated substitution and the law of iterated expectations, we can also write (9) as

$$p_{it} = E_t\left[\sum_{s=t+1}^{\infty} \beta^{s-t} \frac{u'(c_s)}{u'(c_t)}\, y_{is}\right], \tag{177}$$

i.e., the current share price is the expected present discounted value of future dividends, where the discount factors are the intertemporal marginal rates of substitution.

We can also write

$$p_t = \beta\, E_t\Big[p_{t+1}\frac{u'(y_{t+1})}{u'(y_t)}\Big] + \beta\, E_t\Big[y_{t+1}\frac{u'(y_{t+1})}{u'(y_t)}\Big]. \tag{178}$$

The same equation one period forward is

$$p_{t+1} = \beta\, E_{t+1}\Big[p_{t+2}\frac{u'(y_{t+2})}{u'(y_{t+1})}\Big] + \beta\, E_{t+1}\Big[y_{t+2}\frac{u'(y_{t+2})}{u'(y_{t+1})}\Big]. \tag{179}$$

Substituting this into the previous equation gives

$$p_t = \beta\, E_t\left[\beta\, E_{t+1}\Big(p_{t+2}\frac{u'(y_{t+2})}{u'(y_{t+1})}\Big) + \beta\, E_{t+1}\Big(y_{t+2}\frac{u'(y_{t+2})}{u'(y_{t+1})}\Big)\right]\frac{u'(y_{t+1})}{u'(y_t)} + \beta\, E_t\left[y_{t+1}\frac{u'(y_{t+1})}{u'(y_t)}\right]. \tag{180}$$

Rearranging, we obtain

$$p_t = \beta\, E_t\Big[y_{t+1}\frac{u'(y_{t+1})}{u'(y_t)}\Big] + \beta^2\, E_t\Big[y_{t+2}\frac{u'(y_{t+2})}{u'(y_t)}\Big] + \beta^2\, E_t\Big[p_{t+2}\frac{u'(y_{t+2})}{u'(y_t)}\Big]. \tag{181}$$

By the law of iterated expectations, we can eliminate the inner expectation and write:

$$p_t = \beta\, E_t\Big[y_{t+1}\frac{u'(y_{t+1})}{u'(y_t)}\Big] + \beta^2\, E_t\Big[y_{t+2}\frac{u'(y_{t+2})}{u'(y_t)}\Big] + \cdots + \beta^T\, E_t\Big[p_{t+T}\frac{u'(y_{t+T})}{u'(y_t)}\Big]. \tag{182}$$

Since p_t cannot be increasing faster than β^t, we have

$$\lim_{T\to\infty} \beta^T\, E_t\Big[p_{t+T}\frac{u'(y_{t+T})}{u'(y_t)}\Big] = 0. \tag{183}$$

Recall that for any two random variables, $\text{cov}(X,Y) = E(XY) - E(X)E(Y)$, then

$$\text{cov}_t(R_{it}, m_t) + E_t(R_{it})\, E_t(m_t) = 1. \tag{184}$$

Therefore, shares with high expected returns are those for which the covariance of the asset's return with the intertemporal marginal rate of substitution is low. That is, the representative consumer values more an asset which is likely to have high payoffs when aggregate consumption is low.

35.5 The Equity Premium Puzzle

35.5.1 Empirical Observation and Motivation

Mehra and Prescott (1985): The equity premium is an empirical regularity observed in the United States asset markets over the last century. It consists of the difference between the returns on stocks and government bonds. Investors who maintained a portfolio with the same composition as the Standard and Poor's SP500 index would have obtained, if patient enough, a return around 6% higher than those investing all their money in government bonds. Since shares are riskier than bonds, this fact should be explainable by the representative agent's risk aversion.

35.5.2 Modeling Approach

Mehra and Prescott's exercise was intended to confront the theory with observations. They computed statistics of (de-trended) aggregate consumption in the United States and used those statistics to generate an endowment process in their model economy. They calibrated their model to simulate the response of a representative agent to the assumed endowment process. Their results were striking in that the model predicts an equity premium that is significantly lower than the one observed in the United States.

In the Lucas asset pricing framework, the logic is:

$$\text{Dividend process} \to \text{Preferences} \to \text{Prices.}$$

In Mehra-Prescott (1985), a different approach is taken:

$$\text{Consumption process} \to \text{Preferences} \to \text{Prices} \to \text{Returns.}$$

Mehra-Prescott assume that the asset pays off aggregate consumption instead of dividends. That is, returns are given by

$$R_{t+1} = \frac{p_{t+1} + c_{t+1}}{p_t}.$$

Since aggregate consumption is used instead of the dividend, we can substitute c_{t+1} for y_{t+1} to obtain the following equation:

$$p_t = \beta\, E_t\Big[p_{t+1}\frac{u'(c_{t+1})}{u'(c_t)}\Big] + \beta\, E_t\Big[c_{t+1}\frac{u'(c_{t+1})}{u'(c_t)}\Big]. \tag{185}$$

35.5.3 A Two-State Markov Model

Suppose that consumption growth is controlled by λ_t, a random variable which follows a 2-state Markov process:

$$c_{t+1} = \lambda_t c_t. \tag{186}$$

The transition matrix for λ_t is:

$$\phi = \begin{bmatrix} \phi_{11} & (1-\phi_{11}) \\ \phi_{22} & (1-\phi_{22}) \end{bmatrix}, \tag{187}$$

where the states are λ_1 or λ_2, and ϕ_{ij} is the probability of transitioning from state i (current state) to state j (future state).

If the state variable and the probability distribution are known, then we can compute the pricing function, $p(c, i)$, as a function of consumption and the present state. Consider the case with 2 states and CRRA utility. Then from (2) and (3) we have

$$\begin{aligned} p(c,1) = \beta\Big[\phi_{11}\, p(\lambda_1 c, 1)\frac{(\lambda_1 c)^{-\sigma}}{c^{-\sigma}} + (1-\phi_{11})\, p(\lambda_2 c, 2)\frac{(\lambda_2 c)^{-\sigma}}{c^{-\sigma}}\Big] \\ + \beta\Big[\phi_{11}\,\lambda_1 c\,\frac{(\lambda_1 c)^{-\sigma}}{c^{-\sigma}} + (1-\phi_{11})\,\lambda_2 c\,\frac{(\lambda_2 c)^{-\sigma}}{c^{-\sigma}}\Big]. \end{aligned}$$

After simplifying,

$$\begin{aligned} p(c,1) = \beta\Big[\phi_{11}\, p(\lambda_1 c, 1)\,\lambda_1^{-\sigma} + (1-\phi_{11})\, p(\lambda_2 c, 2)\,\lambda_2^{-\sigma}\Big] \\ + \beta\Big[\phi_{11}\,\lambda_1^{1-\sigma} c + (1-\phi_{11})\,\lambda_2^{1-\sigma} c\Big]. \end{aligned} \tag{188}$$

Similarly, one obtains $p(c,2)$.

We guess a linear solution for this functional equation,

$$p(c,i) = w_i c. \tag{189}$$

Thus, the price of the risky asset is linear in consumption c, with the "weight" w_i varying according to the state $i = 1,2$. This is because λ_i only changes the level of consumption. Substituting (6) into the pricing equation, we obtain a system of two equations with two unknowns. This procedure works for any case in which the state space is finite. In general, we have

$$p(c,i) = \beta \sum_{j=1}^{J} \phi_{ij}\, \lambda_j^{-\sigma} \Big[p(\lambda_j c, j) + \lambda_j c\Big] = w_i c, \tag{190}$$

where J is the number of states. In this way, the asset prices depend on the deep parameters of the model: λ_1, λ_2, ϕ_{11}, ϕ_{22}, β, and σ.

35.6 Expected Returns and the Lucas Model

We can now use (1) to get the return. Since the growth of consumption is stochastic then also is the returns function (risky). Note that we are actually solving for expected returns $E[R]$.

$$E[R(i)] = \sum_{j=1}^{2} \phi_{ij} \frac{\lambda_i (w_i + 1)}{w_i} \tag{191}$$

The Lucas model can be used to price a wide class of assets. We introduce a risk free bond (the safe asset). This asset is traded in a competitive market each date. It is a promise to pay one unit of consumption in the following period (it pays for sure!).

We can set up the Lucas model again for the representative consumer by adding this additional asset, b, which has price denoted by p_f. The budget constraint would read as follows:

$$\sum_{i=1}^{n} p_{it} z_{i,t+1} + c_t + p_f b_{t+1} = \sum_{i=1}^{n} z_{it}(p_{it} + y_{it}) + b_t. \tag{192}$$

Note that b_{t+1} can be negative, that means the representative agent can issue bonds or, in other words, borrow.

35.7 Equilibrium Bond Pricing

In equilibrium, we will have $b_t = 0$, i.e. there is a zero net supply of bonds, and prices need to be such that the bond market clears.

We wish to determine p_f, and this can be done by re-solving the consumer's problem, but it is more straightforward to simply use the pricing equation, setting $p_{i,t+1} = 0$ (since these are one-period bonds, they have no value at the end of period $t+1$) and $y_{i,t+1} = 1$. Thus we get that

$$p_f^t = \beta E_t \frac{u'(c_{t+1})}{u'(c_t)}. \tag{193}$$

Given the stochastic process we already assumed, from (10):

$$p_f^t(c,1) = \beta \left[\phi_{11} \frac{(\lambda_1 c)^{-\sigma}}{c^{-\sigma}} + (1-\phi_{11}) \frac{(\lambda_2 c)^{-\sigma}}{c^{-\sigma}} \right] \tag{194}$$

Thus,

$$p_f^t(1) = \beta \left[\phi_{11} \lambda_1^{-\sigma} + (1 - \phi_{11}) \lambda_2^{-\sigma} \right] \tag{195}$$

and

$$R_f(1) = \frac{1}{p_f^t(1)} = ER_f. \tag{196}$$

35.8 Defining the Equity Premium and Calibration

In the model,

$$\text{Equity premium}^{model} = ER_R - ER_f.$$

Empirically, the equity premium is defined as

$$\text{Equity premium}^{empirical} = ER_{S\&P500} - ER_{T\text{-}bill\,(1\,\text{year})}.$$

Note that inflation risk is not in the model but it exists in the actual market. Mehra and Prescott can take the inflation risk out of the model because it is in both $ER_{S\&P500}$ and $ER_{T\text{-}bill}$, so the equity premium is not affected by the presence of inflation risk in the empirical data.

Regarding the calibration, they have that

$$\lambda_1 = 1 + \mu + \delta, \tag{197}$$

$$\lambda_2 = 1 + \mu - \delta, \tag{198}$$

where $\mu = 0.018$ is the average growth rate of consumption in the period studied. As for δ and ϕ, they are chosen to match the standard deviation of consumption growth of 0.036 and its first order serial correlation of -0.14.

For the preference parameters and discount factor, Mehra and Prescott search for (β, σ) from the data by taking a range of possible values:

$$\sigma = [1 \cdots 10] \quad \text{and} \quad \beta = [0.5 \cdots 0.999].$$

They found that for plausible preferences (parameter σ) the model does not produce the equity premium observed in the data. Therefore it is a quantitative puzzle:

Data 1889-1992	Model: $\beta = 0.96,\ \sigma = 2$
$ER_e = 7.8\%$	$ER_e = 7.7\%$
$ER_b = 0.9\%$	$ER_b = 7.4\%$
Equity Premium $= 6.9\%$	Equity premium $= 0.3\%$

The model can only produce the equity premium observed in actual data at the expense of highly unreasonable values of $\beta(> 1)$ or $\sigma(> 10)$.

This incompatibility could be interpreted as evidence against the neoclassical growth model (and related traditions) in general, or as a signal that some of the assumptions used by Mehra and Prescott (profusely utilized in the literature) need to be revised. It is a "puzzle" that the actual behavior differs so much from the predicted behavior, because we believe that the microfoundations tradition is essentially correct and should provide accurate predictions.

36 Heterogeneity in Macroeconomics

36.1 Introduction

In this section we study a class of model economies in which agents (either firms and/or consumers) are heterogeneous in some dimension. We focus on environments where consumers make consumption and savings decisions when they are hit by exogenous and idiosyncratic income shocks. The models considered feature incomplete markets, in which:

- Mutual insurance contracts (A-D markets) are not allowed.
- Only self-insurance is allowed.

The model is constructed around three building blocks:

1. The "income-fluctuation problem"
2. The aggregate neoclassical production function
3. The equilibrium of the asset market.

We focus on the stationary equilibrium and will contrast the results with no insurance and full insurance cases.

The goal is to build a class of models whose equilibria feature a nontrivial endogenous distribution of income and wealth across agents. This allows us to analyze questions such as:

1. How much of the observed wealth inequality can be explained through uninsurable earnings variation across agents?
2. What is the fraction of aggregate savings due to the precautionary motive?
3. What is the welfare cost of incomplete markets?
4. What are the redistributional implications of various policies? How are inequality and welfare affected by such policies?
5. How large are the welfare losses of a rise in labor market risk?

It is key to have an equilibrium model to answer policy questions, because changes in policy affect both the behavior of agents (as highlighted by the Lucas critique) and equilibrium prices (in a general equilibrium framework).

36.2 Aggregation in the Neoclassical Growth Model

36.2.1 Aggregation of Firms

Consider an economy with M firms, indexed by $i = 1, 2, \ldots, M$, which produce a homogeneous good with the same technology

$$zF(k_i, n_i),$$

where z is aggregate productivity. Assume F is strictly increasing, strictly concave, differentiable in both arguments and exhibits constant returns to scale (CRS).

Firm's Problem A price-taking firm maximizes profit:

$$\max_{k_i,n_i}\{zF(k_i,n_i)-(r+\delta)k_i-wn_i\}$$

with first order conditions:

$$zF_k(k_i,n_i)=(r+\delta), \tag{199}$$

$$zF_n(k_i,n_i)=w. \tag{200}$$

Given the CRS assumption, F_k and F_n are homogeneous of degree 0, i.e.,

$$F_k(k_i/n_i,1)\equiv f_k(k_i/n_i) \quad \text{and} \quad F_n(k_i/n_i,1)\equiv f_n(k_i/n_i),$$

so that

$$\frac{F_k(k_i,n_i)}{F_n(k_i,n_i)}=\frac{f_k(k_i/n_i)}{f_n(k_i/n_i)}.$$

Dividing through the first order conditions, we have:

$$\frac{f_k(k_i/n_i)}{f_n(k_i/n_i)}=\frac{(r+\delta)}{w}. \tag{201}$$

Strict monotonicity implies that the function

$$g(k_i/n_i)\equiv\frac{f_k(k_i/n_i)}{f_n(k_i/n_i)}$$

is invertible, so that

$$\frac{k_i}{n_i}=g^{-1}\left(\frac{r+\delta}{w}\right) \tag{202}$$

for every $i=1,2,\ldots,M$. Denote average capital and labor inputs as K and N, respectively, so that:

$$\frac{k_i}{n_i}=\frac{K}{N}. \tag{203}$$

Thus, every firm chooses the same capital-labor ratio.

Returning to the first order condition on capital, we have:

$$zF_k(k_i,n_i)=zf_k\left(\frac{k_i}{n_i}\right)=zf_k\left(\frac{K}{N}\right)=r+\delta.$$

Similarly, we obtain:

$$zF_n(k_i,n_i)=w.$$

This shows the existence of a representative firm with technology $zF(K,N)$.

Aggregation Using Euler's Theorem Aggregating across firms' outputs, by Euler's theorem for CRS production functions, we have:

$$\begin{aligned}\sum_{i=1}^{M}zF(k_i,n_i)&=z\sum_{i=1}^{M}[F_k(k_i,n_i)k_i+F_n(k_i,n_i)n_i]\\&=z\left[f_k\left(\frac{K}{N}\right)\sum_{i=1}^{M}k_i+f_n\left(\frac{K}{N}\right)\sum_{i=1}^{M}n_i\right]\\&=zf_k\left(\frac{K}{N}\right)K+zf_n\left(\frac{K}{N}\right)N=zF(K,N).\end{aligned}$$

36.2.2 Aggregation of Consumers

Consider a version of the neoclassical growth model with N types of consumers, indexed by $i = 1, 2, \ldots, N$, each with the same endowment of capital $k_i^0 = \kappa$ and identical preferences:

$$U(c_0^i, c_1^i, \ldots) = \sum_{t=0}^{\infty} \beta^t u(c_t^i).$$

Markets are competitive, so every consumer faces the same prices. Since all the N consumers make the same decisions, one might aggregate them into a representative agent. However, if the utility function u is not strictly concave, agents may not make the same optimal choices regarding consumption and leisure.

36.2.3 Gorman Aggregation

Suppose there are N consumers indexed by $i = 1, 2, \ldots, N$ and M consumption goods $c \equiv \{c_1, c_2, \ldots, c_M\}$. Consumers have strictly increasing and concave utility functions $u_i : \mathbb{R}^M \to \mathbb{R}$ and face prices $p \equiv \{p_1, p_2, \ldots, p_M\}$. Each consumer is endowed with wealth w_i, and chooses a consumption vector $c^i(p, w_i)$.

Aggregate demand for the vector of goods is given by:

$$c(p, w) = \sum_{i=1}^{N} c^i(p, w_i).$$

Clearly, aggregate demand depends on the distribution of individual wealth. The key question in Gorman aggregation is: Under what conditions can aggregate demand be expressed as a function of aggregate wealth $W = \sum_{i=1}^{N} w_i$ rather than the full wealth distribution?

If we redistribute wealth among the N agents by $dw = (dw_1, \ldots, dw_N)$ such that $\sum_{i=1}^{N} dw_i = 0$, then there is no total change in consumption if:

$$\sum_{i=1}^{N} \frac{\partial c_j^i(p, w_i)}{\partial w_i} dw_i = 0 \quad \forall j = 1, 2, \ldots, M.$$

This will occur if each consumer has the same marginal propensity to consume (MPC) out of wealth:

$$\frac{\partial c_j^i(p, w_i)}{\partial w_i} = \frac{\partial c_j^k(p, w_k)}{\partial w_k} \quad \forall i, k = 1, 2, \ldots, N, \quad \forall j = 1, 2, \ldots, M. \tag{204}$$

A sufficient condition for this is that each consumer's Engel curve is identical and linear:

$$c^i(p, w_i) = a^i(p) + b(p) w_i.$$

Thus, aggregate demand becomes:

$$C(p, w) = \sum_{i=1}^{N} c^i(p, w_i) = \left(\sum_{i=1}^{N} a^i(p) \right) + b(p) W.$$

In general, aggregate consumption can be expressed as a function of aggregate wealth if and only if agents have preferences that admit indirect utility functions of the Gorman form (MWG Prop. 4.B.1). This result shows how to construct the preferences of a representative consumer by aggregating the preferences of individual agents, although it requires strong restrictions on the utility functions.

36.2.4 Concluding Remarks on Aggregation

If every firm has the same CRS production function, and if consumers have the same initial endowments and identical, strictly concave preferences, then the neoclassical growth model admits a formulation with one representative firm and one representative household.

37 The Neoclassical Growth Model (NGM) with Heterogeneity in Endowments

37.1 Overview and Demographics

Consumers are differentiated only by their initial endowments of wealth. The economy is populated by N types of infinitely lived agents, indexed by $i = 1, 2, \ldots, N$, with corresponding measures μ_i normalized such that $\sum_{i=1}^{N} \mu_i = 1$. Preferences are time separable and defined over streams of consumption:

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t^i).$$

The utility function belongs to one of the following classes (log, power, or exponential), allowing for a subsistence level $\bar{c}$:

$$\begin{aligned} u(c) &= \ln(\bar{c} + c), \quad \bar{c} + c \geq 0, \\ u(c) &= (\bar{c} + c)^{\sigma}, \quad \bar{c} + c \geq 0, \\ u(c) &= -\bar{c} \exp(-\eta c). \end{aligned}$$

These utility functions yield linear Engel curves in wealth; that is, any given change in wealth induces the same change in consumption, independently of the wealth level.

37.2 Technology and the Firm's Problem

The aggregate production technology is given by:

$$y_t = f(K_t),$$

with f strictly increasing, strictly concave, and differentiable. The representative firm chooses investment I_t to maximize the present value of future profits:

$$A_t = \max_{I_t} \sum_{\tau=t}^{\infty} \frac{p_\tau}{p_t} [f(K_\tau) - I_\tau], \tag{205}$$

subject to the law of motion for capital:

$$K_{\tau+1} = (1 - \delta) K_\tau + I_\tau. \tag{206}$$

Here, p_t denotes the time t price of the final good. Defining real profits as $\pi_t \equiv f(K_t) - I_t$, A_t represents the firm's value, i.e. the present value of future profits discounted by the relative price of consumption between periods.

37.3 Markets and the Household Problem

37.3.1 Market Structure

There are spot markets for the final good and complete financial markets, meaning there are no constraints on transfers of income across periods.

37.3.2 The Household's Optimization Problem

The maximization problem for household i at time t is:

$$\max_{\{c_t^i\}} \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau^i) \tag{207}$$

subject to the budget constraint:

$$\sum_{\tau=t}^{\infty} p_\tau c_\tau^i \leq p_t a_t^i, \tag{208}$$

where a_t^i represents the wealth of agent i in consumption units. In general, the wealth of agent i at time t can be defined as:

$$p_t a_t^i = s_t^i A_t, \tag{209}$$

with s_t^i being the share of the firm's value owned by consumer i at time t. Summing over all agents and using $\sum_{i=1}^{N} \mu_i s_t^i = 1$ for every t, we obtain that aggregate household wealth equals the value of the firm, A_t.

37.3.3 Solving the Household Problem (Log Preferences)

Assume log preferences. The first-order condition (FOC) of the household at time t is:

$$u'(c_\tau^i)\beta^\tau = \lambda_t^i p_\tau, \tag{210}$$

which, for log utility ($u(c) = \ln(\bar{c} + c)$), becomes

$$\frac{1}{\bar{c} + c_\tau^i}\beta^\tau = \lambda_t^i p_\tau. \tag{211}$$

Thus, the consumption function is given by:

$$c_\tau^i = \frac{\beta^\tau}{\lambda_t^i p_\tau} - \bar{c} \tag{6}$$

where λ_t^i is the Lagrange multiplier on the time t budget constraint. Substituting this expression into the budget constraint yields:

$$\sum_{\tau=t}^{\infty} p_\tau \left(\frac{\beta^\tau}{\lambda_t^i p_\tau} - \bar{c} \right) = p_t a_t^i, \tag{212}$$

$$\frac{1}{\lambda_t^i}(1-\beta) - \bar{c} \sum_{\tau=t}^{\infty} p_\tau = p_t a_t^i, \tag{213}$$

$$\frac{1}{\lambda_t^i} = (1-\beta) p_t a_t^i + (1-\beta)\bar{c} \sum_{\tau=t}^{\infty} p_\tau. \tag{7}$$

Substituting (7) back into (6) for $\tau = t$, we obtain:

$$c_t^i = \frac{1}{p_t} \left[(1-\beta) p_t a_t^i + (1-\beta)\bar{c} \sum_{\tau=t}^{\infty} p_\tau \right] - \bar{c}, \tag{214}$$

$$c_t^i = \bar{c} \left[(1-\beta) \sum_{\tau=t}^{\infty} \left(\frac{p_\tau}{p_t} \right) - 1 \right] + (1-\beta) a_t^i, \tag{215}$$

$$c_t^i = \Theta(p_t, \bar{c}) + (1-\beta) a_t^i, \tag{8}$$

where $\Theta(p_t, \bar{c})$ is a function of the subsistence level and the sequence of prices $\{p_t, p_{t+1}, \ldots\}$. This shows that the optimal consumption choice at time t is an affine function of asset holdings for each agent. Although derived for the log-case, a similar representation holds for power and exponential utility functions.

The first consequence of equation (8) is that aggregate consumption depends only on aggregate variables (prices and aggregate wealth):

$$C_t = \Theta(p_t, \bar{c}) + (1 - \beta)A_t. \tag{9}$$

Here, A_t can be expressed as a function of the sequence of prices $\{p_\tau\}_{\tau=t}^{\infty}$ and aggregate capital stocks $\{K_t\}_{\tau=t}^{\infty}$. Consequently, the competitive equilibrium can be recovered from the following standard single-agent problem:

$$\max_{\{C_\tau\}} \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(C_\tau) \tag{216}$$

subject to

$$\sum_{\tau=t}^{\infty} p_\tau C_\tau \leq p_t A_t, \tag{217}$$

with FOC's given by

$$\frac{u'(C_t)}{u'(C_{t+1})} = \frac{p_t}{p_{t+1}}. \tag{218}$$

37.4 Implications for Aggregate Dynamics

The FOC of the representative firm's problem is:

$$p_t = p_{t+1}\left[f'(K_{t+1}) + (1 - \delta)\right]. \tag{219}$$

Combining this with the household FOC, we obtain the Euler Equation:

$$u'(C_t) = \beta u'(C_{t+1})\left[f'(K_{t+1}) + (1 - \delta)\right]. \tag{220}$$

37.5 Steady State

In steady state, we have $f'(K^*) = \frac{1}{\beta} - (1 - \delta)$ and $p_t/p_{t+1} = 1/\beta$. Using the definition of $\Theta(p_t, \bar{c})$, it follows that:

$$c^i = (1 - \beta)a^i. \tag{1}$$

Thus, in steady state, the average propensity to save is β, independent of wealth, for every household type.

37.6 Equilibrium Dynamics of the Wealth Distribution

Starting from the lifetime budget constraint of agent i at time t:

$$p_t c_t^i + \sum_{\tau=t+1}^{\infty} p_\tau c_\tau^i = p_t a_t^i, \tag{221}$$

we have

$$p_t c_t^i + p_{t+1} a_{t+1}^i = p_t a_t^i, \tag{222}$$

$$\frac{c_t^i}{a_t^i} + \frac{p_{t+1} a_{t+1}^i}{p_t a_t^i} = 1, \tag{223}$$

$$\frac{a_{t+1}^i}{a_t^i} = \frac{p_t}{p_{t+1}} \left(1 - \frac{c_t^i}{a_t^i}\right). \tag{2}$$

This equation expresses the growth rate of wealth for type i as a function of the consumption-to-wealth ratio.

By aggregating over types, we obtain:

$$\frac{A_{t+1}}{A_t} = \frac{p_t}{p_{t+1}} \left(1 - \frac{C_t}{A_t}\right). \tag{3}$$

Furthermore, from Equation (8) we have:

$$\frac{c_t^i}{a_t^i} = \frac{\Theta(p_t, \bar{c})}{a_t^i} + (1 - \beta), \tag{4}$$

$$\frac{C_t}{A_t} = \frac{\Theta(p_t, \bar{c})}{A_t} + (1 - \beta). \tag{5}$$

Together with equation (2), these imply:

$$\frac{a_{t+1}^i}{a_t^i} > \frac{A_{t+1}}{A_t} \tag{6}$$

if and only if

$$\frac{c_t^i}{a_t^i} < \frac{C_t}{A_t} \tag{7}$$

if and only if

$$\Theta(p_t, \bar{c})(a_t^i - A_t) > 0 \tag{8}$$

if and only if

$$\frac{s_{t+1}^i}{s_t^i} > 1. \tag{9}$$

Thus, the evolution of consumer i's wealth share over time is determined by the sign of the constant Θ (which is the same for all agents) and the agent's position relative to the aggregate. In the absence of a subsistence level (i.e., $\bar{c} = 0$ and $\Theta = 0$), the neoclassical growth model with heterogeneous endowments predicts that the wealth distribution remains unchanged along the transition path—initial inequality persists indefinitely.

In the presence of a subsistence level, the dynamics change. For example, if $\Theta > 0$ and $a_t^i > A_t$, then consumer i's wealth share will grow over time, leading to increased inequality. The following lemma clarifies this point:

Lemma (Chaterjee, JPubE, 1994): The common constant term of the consumption function $\Theta(p_t, \bar{c})$ is greater (less) than zero if and only if the economy is converging from below (above) to the steady state.

38 Aggregation with Complete Markets: Nigishi's Approach

38.1 Overview

In economies that do not admit Gorman aggregation, we may still recover a representative consumer under alternative assumptions on primitives. Nigishi (1960) proposes a method to compute the competitive equilibrium prices and allocations of complete markets economies with heterogeneous households for which the welfare theorems hold. If the First Welfare Theorem holds, then any competitive equilibrium is Pareto efficient. Therefore a competitive equilibrium of an economy with heterogeneous agents can be found as a solution to the Social Planner's Problem with the correctly chosen Pareto weights assigned to each agent.

38.2 Nigishi's Approach

Consider an infinite horizon economy with consumers and firms. Suppose there are two consumers ($N = 2$) that are endowed with identical preferences given by $u : \mathbb{R}_+ \to \mathbb{R}$ that are strictly increasing, strictly concave, and twice continuously differentiable. Consumers are endowed with initial wealth $\{a_0^1, a_0^2\}$ and face a sequence of given prices $p \equiv \{p_t\}_{t=0}^{\infty}$. Consumers trade assets in complete markets, so that there are no constraints on intertemporal transfers (e.g., no borrowing constraints on debt).

The consumer problem reads

$$v(p, a_0^i) = \max_{\{c_t^i\}_{t=0}^{\infty}} \left[\sum_{t=0}^{\infty} \beta^t u(c_t^i) \right] \tag{224}$$

subject to

$$\sum_{t=0}^{\infty} p_t c_t^i \leq p_0 a_0^i. \tag{225}$$

Let λ_i be the multiplier on the budget constraint for agent $i \in \{1, 2\}$. Taking the first order conditions (FOC's) we have

$$\beta^t u'(c_t^i) = \lambda_i p_t \tag{11}$$

Thus,

$$\frac{u'(c_t^1)}{u'(c_t^2)} = \frac{\lambda_1}{\lambda_2} \tag{12}$$

Suppose there is a representative firm that owns physical capital and makes investment decisions. The firm is endowed with k_0 units of capital at time 0. The household owns the shares in the firm and therefore the firm uses the consumers' wealth to finance investments. Denote the time-0 value of the firm by A_0, and note that $a_0^1 + a_0^2 = A_0$. The present value of the firm's profits is:

$$A_0 = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \left[\sum_{t=0}^{\infty} \left(\frac{p_t}{p_0} \right) (f(k_t) + (1 - \delta)k_t - k_{t+1}) \right], \tag{226}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, strictly concave, and differentiable.

The firm's FOC is:

$$1 = \frac{p_{t+1}}{p_t} (f'(k_{t+1}) + (1 - \delta)). \tag{227}$$

Suppose we have the optimal sequence of capital stock, $k^?_{t+1}$, then we can write the value of the firm recursively as

$$A_t = (f(k^?_t) + (1-\delta)k^?_t - k^?_{t+1}) + \frac{p_t}{p_0}A_{t+1}. \tag{228}$$

We can also rearrange the consumer's budget constraint:

$$p_t c^i_t + \sum_{\tau>t} p_\tau c^i_\tau = p_t a^i_t, \tag{229}$$

$$p_t c^i_t + p_{t+1} a^i_{t+1} = p_t a^i_t, \tag{230}$$

$$c^i_t + \frac{p_{t+1}}{p_t} a^i_{t+1} = a^i_t. \tag{231}$$

Summing over i:

$$\sum_{i=1}^{2} c^i_t + \frac{p_{t+1}}{p_t}\sum_{i=1}^{2} a^i_{t+1} = \sum_{i=1}^{2} a^i_t, \tag{232}$$

$$C_t + \frac{p_{t+1}}{p_t} A_{t+1} = A_t. \tag{233}$$

By substituting the firm's value we get that

$$C_t + k^?_{t+1} = f(k^?_t) + (1-\delta)k^?_t. \tag{234}$$

38.3 Nigishi's Planner

Now consider the Negishi Planner's problem of choosing allocations to maximize social welfare subject to the resource constraint:

$$v(\mu_1, \mu_2, k_0) = \max_{\{c^1_t, c^2_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} [\mu_1 u(c^1_t) + \mu_2 u(c^2_t)] \tag{235}$$

subject to

$$c^1_t + c^2_t + k_{t+1} = f(k_t) + (1-\delta)k_t, \tag{236}$$

where k_0 is given and μ_i are the planner's weights for each consumer.

Taking the first order conditions we have

$$\mu_i u'(c^i_t) = \lambda^{RC}_t, \tag{237}$$

$$u'(c^i_t) = \beta u'(c^i_{t+1})(f'(k_{t+1}) + (1-\delta)) \tag{13}$$

Thus,

$$\frac{u'(c^1_t)}{u'(c^2_t)} = \frac{\mu_2}{\mu_1} \tag{14}$$

Equation (14) tells us that (i) the planner keeps their relative marginal utilities of consumption constant, and (ii) consumption is allocated to each consumer proportionately to its planner weights.

Implementing the Planner's Allocation: One may now select the Negishi weights so that the planner implements the competitive equilibrium allocation. Combining the competitive equilibrium allocation (equation (12)) and the Negishi Planner's allocation (equation (14)) we obtain a condition on the Negishi weights:

$$\frac{\lambda_2}{\lambda_1} = \frac{\mu_2}{\mu_1} \tag{15}$$

Given the same initial capital, these Negishi weights will also ensure that competitive equilibrium prices induce the same path for investment under both competitive and centralized allocations. This follows from a straightforward comparison of the competitive firm's first order condition and the Negishi planner's intertemporal optimality condition:

$$p_t = \beta p_{t+1}(f'(k_{t+1}) + 1 - \delta), \tag{238}$$

$$\lambda_t^{RC} = \beta \lambda_{t+1}^{RC}(f'(k_{t+1}) + 1 - \delta). \tag{239}$$

Therefore the competitive equilibrium prices can be obtained from the sequence of multipliers on the Negishi planner's resource constraint:

$$\frac{\lambda_t^{RC}}{\lambda_{t+1}^{RC}} = \frac{p_t}{p_{t+1}} \tag{16}$$

Intuitively, the multiplier on the resource constraint is the marginal value of an extra unit of consumption. In competitive equilibrium, the price is also the marginal value of an additional unit of consumption.

39 Income Fluctuation Problem

39.1 Introduction

The set of models in this section are constructed around three building blocks:

1. the "income-fluctuation problem",
2. the aggregate neoclassical production function, and
3. the equilibrium of the asset market.

We focus on the stationary equilibrium and we will contrast the results with no insurance and full insurance.

Consider an individual subject to exogenous income shocks. These shocks are not fully insurable because of the lack of a complete set of Arrow-Debreu contingent claims. We assume that there is only a risk-free asset (i.e., with a fixed rate of return) in which the individual can save/borrow, and that the individual faces an exogenously set borrowing (liquidity) constraint.

There are two reasons for savings: intertemporal substitution and the precautionary motive, the latter serving as a self-insurance strategy to hedge against low earnings in the future. Individuals who bear a long sequence of bad shocks will have low wealth and will be close to the constraint, while those with a long run of good shocks will have high wealth. A continuum of such agents subject to different shocks will give rise to a wealth distribution. Integrating wealth holdings across all agents will give rise to an aggregate supply of capital.

39.1.1 Aggregate Production Function and Asset Market Equilibrium

Profit maximization of the competitive representative firm operating a constant returns to scale (CRS) technology gives rise to an aggregate demand for capital. When the demand and supply interact in an asset market, an equilibrium interest rate arises endogenously. Notice that if a full set of Arrow-Debreu contingent claims were available, the economy would collapse to a representative agent model with a stationary amount of savings such that $(1+r)\beta = 1$. With uninsurable risk, the supply of savings is larger (so that r is lower) because of precautionary reasons, and consequently $(1+r)\beta < 1$. In this model economy, agents can only trade an asset with non state-contingent payoffs, so financial markets are incomplete. This is not the only possible form of market incompleteness, but it is a good starting point to think about how the lack of insurance markets affects equilibrium allocations compared to the Pareto efficient outcome. Later, models with a deeper source of market incompleteness will be discussed.

39.2 Outline of the Models

1. Setup of the Model,
2. Two extremes: Autarky and Complete Markets,
3. Income Fluctuation Problem with no uncertainty,
4. Income Fluctuation Problem with IID shocks, and
5. Income Fluctuation Problem with Markov shocks.

39.3 Setup of the Model

There is a continuum of ex ante identical consumers. Each consumer receives an exogenous stochastic idiosyncratic income shock each period, so that ex-post they are heterogeneous.

Preferences:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{240}$$

where u is strictly increasing, strictly concave, differentiable, with $u'(0) = +\infty$ and $\lim_{c\to\infty} u'(c) = 0$.

Income Process: Income in each period can take values (in increasing order)

$$y \in Y = \{\bar{y}_1, \bar{y}_2, ..., \bar{y}_s\}, \tag{241}$$

with the probability of receiving income $\bar{y}_s$ equal to $\bar{\pi}_s$ for each $s \in S$, where $\sum_{s\in S} \bar{\pi}_s = 1$. The aggregate endowment is

$$Ey = \sum_{s\in S} \bar{\pi}_s \bar{y}_s. \tag{242}$$

Income Histories: The history of incomes up to period t is

$$y^t = \{y_0, y_1, ..., y_t\} \in Y^{t+1}, \tag{243}$$

with the probability

$$\pi_t(y^t) : \sum_{y^t \in Y^{t+1}} \pi_t(y^t) = 1. \tag{244}$$

Consumption Allocation: The consumption allocation is given by

$$c = \{c_t(y^t)\}, \quad \forall t \geq 0, \forall y^t \in Y^{t+1}. \tag{245}$$

Different market and/or insurance arrangements will lead to a different c.

39.4 Income Fluctuation Problem: Autarky

In autarky, no savings or other insurance arrangements are possible. The simple solution is:

$$c_t^{AUT}(y^t) = y_t, \quad \forall t \geq 0, \forall y^t \in Y^{t+1}, \tag{246}$$

which implies a perfect correlation between an individual's income and consumption.

39.5 Income Fluctuation Problem: Complete Markets

39.5.1 Problem Formulation

Agents maximize expected utility:

$$\max_c \sum_{t=0}^{\infty} \sum_{y^t \in Y^t} \beta^t u(c_t(y^t))\pi_t(y^t) \tag{1}$$

subject to the Arrow-Debreu structure budget constraint:

$$\sum_{t=0}^{\infty} \sum_{y^t \in Y^t} p_t(y^t)c_t(y^t) \leq \sum_{t=0}^{\infty} \sum_{y^t \in Y^t} p_t(y^t)y_t \tag{2}$$

where $p_t(y^t)$ is the price of consumption in state y^t in terms of time 0 consumption. Denote the price system by $p = \{p_t(y^t)\}$.

39.5.2 Equilibrium

Definition: An Arrow-Debreu equilibrium consists of an allocation c and a price system p such that:

1. c maximizes (1) subject to (2) given p, and
2. c is feasible, i.e.,

$$\sum_{y^t \in Y^t} \pi_t(y^t)c_t(y^t) = Ey \quad \forall t \geq 0. \tag{247}$$

39.5.3 Solution

The first order condition for $c_t(y^t)$ is:

$$\beta^t \pi_t(y^t)u'(c_t(y^t)) = \lambda p_t(y^t). \tag{248}$$

Guessing $p_t(y^t) = \beta^t \pi_t(y^t)$ leads to

$$u'(c_t(y^t)) = \lambda, \tag{249}$$

which implies that consumption is constant over time and states of the world. Substituting into the resource constraint yields:

$$c_t^{CM}(y^t) = Ey. \tag{250}$$

Thus, (c^{CM}, p) constitutes an AD equilibrium.

39.5.4 Properties of the Consumption Process

Under complete markets,

1. Consumption $c_t^{CM}(y^t)$ is independent of y^t (full insurance), and
2. Consumption is equal for all agents, i.e.,

$$c_t^{CM}(y^t) = C, \tag{251}$$

where $C = Ey$ is aggregate consumption.

39.5.5 Social Planner's Problem

By the First Welfare Theorem, competitive equilibrium is Pareto optimal and equilibrium allocations can be obtained as the solution to the social planner's problem.

Social Planner's Problem:

$$\max_c \sum_{t=0}^{\infty} \sum_{y^t \in Y^t} \beta^t u(c_t(y^t))\pi_t(y^t) \tag{3}$$

subject to the budget constraint

$$\sum_{y^t \in Y^t} \pi_t(y^t) c_t(y^t) = Ey \tag{4}$$

Let θ_t be the Lagrange multiplier on the resource constraint in period t. The first order conditions give

$$\pi_t(y^t)u'(c_t(y^t)) = \theta_t \pi_t(y^t), \tag{252}$$

implying

$$u'(c_t(y^t)) = \theta_t. \tag{253}$$

Thus,

$$c_t^{PO} = Ey, \tag{254}$$

so that

$$c_t^{PO} = c_t^{CM}. \tag{255}$$

39.6 Summary of Consumption Dynamics

Autarky predicts perfect correlation of consumption and income, whereas complete markets (full insurance) predict zero correlation. Consider the regression:

$$\Delta \ln c_t = \beta_0 + \beta_1 \Delta \ln y_t + \beta_2 \Delta \ln C_t + \varepsilon_t. \tag{256}$$

Under autarky: $\beta_1 = 1$ and $\beta_2 = 0$. Under complete markets (full insurance): $\beta_1 = 0$ and $\beta_2 = 1$.

39.7 Income Fluctuation Problem with No Uncertainty

Without uncertainty, insurance is studied for illustrative purposes. It is instructive to study the consumption decisions of individuals with an uneven income stream who face a borrowing constraint. There is a single risk-free asset a_t in the economy that yields a net rate of return r, where r is exogenous, constant over time, and $\beta(1+r) \leq 1$. Only self-insurance is allowed. Assume that the income sequence $\{y_0, y_1, y_2, \ldots\}$ is deterministic.

Budget Constraint:

$$a_{t+1} + c_t \leq y_t + (1+r)a_t. \tag{257}$$

Define "cash-in-hands" as

$$x_t = y_t + (1+r)a_t. \tag{258}$$

Rewriting the budget constraint in terms of x_t, we have:

$$x_{t+1} = (x_t - c_t)(1+r) + y_{t+1}. \tag{259}$$

A borrowing constraint is imposed:

$$a_{t+1} \geq \bar{a}_{t+1}, \tag{260}$$

or equivalently,

$$x_{t+1} \geq y_{t+1} + (1+r)\bar{a}_{t+1}. \tag{261}$$

39.7.1 Borrowing Constraints: Two Extremes

1. **No-Borrowing Constraint:**

$$a_{t+1} \geq 0 \quad \Leftrightarrow \quad x_{t+1} \geq y_{t+1}. \tag{262}$$

2. **Natural Borrowing Constraint:**
   Consumers cannot borrow more than what they are able to repay. If the consumer borrows the maximum, his consumption in all future periods is zero: $c_{t+\tau} = 0$ for all $\tau \geq 0$, so that

$$a_{t+1} \geq \bar{a}_{t+1} = -\sum_{\tau=1}^{\infty} \frac{y_{t+\tau}}{(1+r)^{\tau}}, \tag{263}$$

   if and only if

$$x_{t+1} \geq \bar{x}_{t+1} = -\sum_{\tau=1}^{\infty} \frac{y_{t+\tau+1}}{(1+r)^{\tau}}. \tag{264}$$

   The natural borrowing constraint will never bind because households will never choose to be in an asset position that may induce, with positive probability, a future state where they end up with zero consumption. This borrowing constraint is directly implied by the condition $c_t > 0$ (Inada, $\lim_{c\to 0} u(c) = -\infty$) holding in every state of the world.

39.8 Consumer's Problem

The consumer's problem is formulated as

$$\max_{c_t, x_{t+1}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{265}$$

subject to

$$x_{t+1} = (x_t - c_t)(1+r) + y_{t+1}, \tag{266}$$
$$x_{t+1} \geq y_{t+1} + (1+r)\bar{a}_{t+1}, \tag{267}$$
$$x_0 = y_0. \tag{268}$$

39.8.1 Recursive Formulation

The consumer's problem can also be written recursively as:

$$v_t(x_t) = \max_{c_t, a_{t+1} \geq \bar{a}_{t+1}} \{u(c_t) + \beta v_{t+1}(x_{t+1})\} \tag{269}$$

subject to

$$c_t + \bar{a}_{t+1} \leq x_t, \tag{270}$$

$$x_{t+1} = (1+r)a_{t+1} + y_{t+1}, \tag{271}$$

or equivalently,

$$v_t(x_t) = \max_{c_t \leq x_t - \bar{a}_{t+1}} \{u(c_t) + \beta v_{t+1}[(x_t - c_t)(1+r) + y_{t+1}]\}. \tag{272}$$

One can show that v_t is strictly increasing, strictly concave, and differentiable.

39.8.2 First Order Conditions and Envelope Condition

The first order condition (FOC) is:

$$u'(c_t) \geq \beta(1+r)v'(x_{t+1}), \tag{273}$$

which holds with equality if $x_{t+1} > y_{t+1} + (1+r)\bar{a}_{t+1}$ (i.e., when the borrowing constraint is not binding).

The envelope condition is given by:

$$v_t'(x_t) = u'(c_t). \tag{274}$$

Combining these conditions, we obtain

$$u'(c_t) \geq \beta(1+r)u'(c_{t+1}), \tag{275}$$

which again holds with equality if the borrowing constraint is not binding.

39.9 Conclusion

These notes have provided an overview of aggregation with complete markets following Nigishi's approach and examined several variations of the income fluctuation problem. We contrasted autarkic outcomes with complete market outcomes, and finally, we analyzed the consumer's decision problem under a deterministic income stream with a borrowing constraint. This framework lays the foundation for further study of stochastic income processes and market incompleteness.

40 Income Fluctuation Problems under No Uncertainty

40.1 Natural Borrowing Constraint with $\beta(1+r) = 1$

If $\beta(1+r) = 1$, then the Euler equation (EE) is

$$u'(c_t) = u'(c_{t+1}). \tag{276}$$

This implies complete consumption smoothing:

$$c_{t+1} = c_t \equiv c^*. \tag{277}$$

From the budget constraint we have

$$c^* = rW_t = r(a_t + H_t), \tag{278}$$

where

$$H_t = \frac{1}{1+r} \sum_{j=0}^{\infty} \frac{y_{t+j}}{(1+r)^j}$$

represents human (non-financial) wealth.

This is a version of the permanent income hypothesis: a permanent income is the annuity value of one's wealth $W_t \equiv a_t + H_t$, and individuals choose consumption equal to their permanent income. Savings are given by

$$s_t = y_t - rH_t,$$

so that if current income deviates from the permanent non-financial wealth, the difference is saved.

Experiments:

- A permanent change in income: reduce income by X every year. Then $\Delta c^* = X$.
- A transitory change in income: reduce income by X only in period t. Then $\Delta c^* = \frac{r}{1+r} X$. Tiny!

40.2 No-Borrowing Constraint with $\beta(1+r) = 1$

Under a no-borrowing constraint, the EE becomes

$$u'(c_t) \geq u'(c_{t+1}), \tag{279}$$

with equality if $x_{t+1} > y_{t+1}$. That is,

$$c_{t+1} = c_t \quad \text{if } x_{t+1} > y_{t+1}, \tag{280}$$

$$c_{t+1} > c_t \quad \text{if } x_{t+1} = y_{t+1} \Leftrightarrow c_t = x_t. \tag{281}$$

Thus, consumption is a nondecreasing sequence. If the agent is not borrowing constrained (i.e., the constraint is not binding) then consumption is constant, whereas if the agent is borrowing constrained, consumption increases.

Intuitively, a declining consumption sequence can be improved by cutting a unit of consumption at t (incurring a loss of $u'(c_t)$) and increasing consumption at $t+1$ by saving and receiving a return of $\beta(1+r)u'(c_{t+1}) = u'(c_{t+1}) > u'(c_t)$, given that u is strictly concave and $c_t > c_{t+1}$. A symmetric argument rules out $c_t < c_{t+1}$ as long as the constraint is not binding; hence, consumption strictly increases only for constrained agents with zero savings or wealth equal to zero.

Define the limiting present value of human wealth as:

$$\bar{H} = \sup_t H_t. \tag{282}$$

Theorem:

$$\bar{c} = \lim_{t\to\infty} c_t = r\bar{H}. \tag{283}$$

The borrowing constraint stops binding when the permanent non-financial wealth reaches its maximum, and consumption thus converges to a constant. Intuition:

- If $\bar{c} > r\bar{H}$ then, ultimately, consumption will not be feasible.
- If $\bar{c} < r\bar{H}$ then consumption is not optimal since the budget constraint is slack.

41 Income Fluctuation Problems with I.I.D. Shocks

41.1 Natural Borrowing Constraint and $\beta(1+r)=1$

In the presence of i.i.d. shocks, the value function becomes

$$v_t(x_t) = \max_c \left\{ u(c) + \beta \sum_{s\in S} \bar{\pi}_s \, v\Big[(x_t - c_t)(1+r) + \bar{y}_s\Big] \right\}. \tag{284}$$

The first-order condition (FOC) is then

$$u'(c) = \beta(1+r) \sum_{s\in S} \bar{\pi}_s \, v'\Big[(1+r)(x-c) + \bar{y}_s\Big], \tag{285}$$

and the envelope condition gives

$$v'(x) = u'(c). \tag{286}$$

Combining these results yields

$$v'(x) = \beta(1+r) \sum_{s\in S} \bar{\pi}_s \, v'(x'_s). \tag{287}$$

41.2 Precautionary Savings: Effects of Uncertainty

Suppose u is three times differentiable. The agents are said to be *prudent* if the marginal utility of consumption is convex, i.e., if

$$u''' > 0.$$

Risk aversion alone does not explain the saving behavior under uncertainty; the third derivative of the utility function is also crucial. Kimball (1990) defines the *coefficient of prudence* as

$$\eta(c) = -\frac{u'''}{u''}. \tag{288}$$

Precautionary savings occur if and only if consumers are prudent.

41.3 Second Order Approximation and the Random Walk

For unconstrained agents the Euler equation implies

$$u'(c) = E\,u'(c'). \tag{289}$$

Taking a second order Taylor approximation of $u'(c')$ around c, we obtain

$$u'(c') = u'(c) + u''(c)(c'-c) + \frac{1}{2}u'''(c)(c'-c)^2. \tag{290}$$

Substituting this into the Euler equation yields:

$$u'(c) = u'(c) + u''(c)(E\,c'-c) + \frac{1}{2}u'''(c)E\left[(c'-c)^2\right], \tag{291}$$

$$E\,c' - c = \eta(c)\frac{\sigma_c^2}{2}, \tag{292}$$

where $\sigma_c^2 = E\left[(c'-c)^2\right]$. Thus,

$$E\,c' = c + \eta(c)\frac{\sigma_c^2}{2}. \tag{293}$$

If agents are prudent, expected future consumption exceeds current consumption. The magnitude of this effect depends on the coefficient of prudence and the variance of consumption. In the absence of uncertainty ($\sigma_c^2 = 0$), if $\eta(c) = 0$ (as in the case of quadratic utility), then consumption follows a random walk (Hall, 1978):

$$E_t c_{t+1} = c_t. \tag{294}$$

42 Prudence and Precautionary Savings Examples

42.1 A Two-Period Model

Consider a simple two-period version of the income fluctuation problem:

$$\max_{c_0,c_1,a_1} \quad u(c_0) + \beta E[u(c_1)] \tag{295}$$

$$\text{s.t.} \quad c_0 + a_1 = y_0, \tag{296}$$

$$c_1 = (1+r)a_1 + \tilde{y}_1, \tag{297}$$

where y_0 is given and $\tilde{y}_1$ is exogenous but stochastic. For algebraic simplicity, assume $\beta(1+r) = 1$. Then the Euler equation is

$$u'(y_0 - a_1) = E\left[u'\Big((1+r)a_1 + \tilde{y}_1\Big)\right]. \tag{298}$$

Since there is one equation and one unknown (a_1), and because the left-hand side is increasing in a_1 (due to $u'' < 0$) while the right-hand side is decreasing (the sum of decreasing functions is decreasing), the optimal asset choice a_1^* is uniquely determined.

42.2 Impact of Increased Uncertainty

Now consider what happens to optimal consumption at $t = 0$ if uncertainty over future income increases, i.e., if $\tilde{y}_1$ becomes more risky. Suppose we introduce a mean preserving spread by letting ε be a random variable with zero mean and positive variance, and define

$$\hat{y}_1 = \tilde{y}_1 + \varepsilon.$$

Then the right-hand side of the Euler equation becomes

$$E\left[u'\Big((1+r)a_1 + \tilde{y}_1 + \varepsilon\Big)\right] = E\left(E\left[u'\Big((1+r)a_1 + \tilde{y}_1 + \varepsilon\Big)\,\Big|\,\tilde{y}_1\right]\right). \tag{299}$$

Using Jensen's inequality and the convexity of u', we obtain

$$E\left(E\left[u'\Big((1+r)a_1 + \tilde{y}_1 + \varepsilon\Big)\,\Big|\,\tilde{y}_1\right]\right) \geq E\left[u'\Big((1+r)a_1 + \tilde{y}_1\Big)\right]. \tag{300}$$

Since $E(\varepsilon) = 0$, this result shows that a mean-preserving spread of $\tilde{y}_1$ increases the marginal value of future resources, which shifts the optimal asset choice upward (i.e., a_1^* increases) and leads to a reduction in c_0.

42.3 Discussion on Prudence

Prudence is characterized by the convexity of the marginal utility, i.e., $u''' > 0$. In fact, any utility function with decreasing absolute risk aversion (DARA) displays a positive third derivative (e.g., CRRA utility). Intuitively, an increase in uncertainty reduces the certainty-equivalent income next period and, with DARA, effectively increases the degree of risk aversion of the agent, inducing additional savings. Such savings are termed "precautionary savings" or "self-insurance." In the two-period model, one may define *precautionary wealth* as the difference between the optimal asset choice under uncertainty, a_1^*, and the asset choice that would prevail if future income were known with certainty.

43 Computational Methods for Income-Fluctuation Problems

43.1 Approximating an i.i.d. Normal Shock

Consider the problem:

$$V(z,a) = \max_{c\geq 0,\, a'\geq \bar{a}} \Big\{u(c) + \beta EV(z',a')\Big\} \tag{301}$$

subject to

$$a(1+r) + we^{z} = a' + c, \quad z' \sim \text{i.i.d. } N(0,\sigma_z^2), \tag{302}$$

with given r and w. To solve this problem by discretizing the state space, we need to approximate the i.i.d. normal shock by a discrete shock.

Let $\{z_i\}_{i=1}^n$ and $\{p_i\}_{i=1}^n$ be the discrete values and associated probabilities. The algorithm is as follows:

1. Set n, the number of realizations.
2. Choose the upper bound $\bar{z}$ and lower bound $\underline{z}$ using a parameter λ, so that

$$\bar{z} = \mu + \lambda\sigma_z, \tag{303}$$

$$\underline{z} = \mu - \lambda\sigma_z. \tag{304}$$

3. Set $\{z_i\}_{i=1}^n$ equally spaced such that

$$z_i = \underline{z} + \frac{2\lambda\sigma_z}{n-1}(i-1), \quad i = 1,2,\ldots,n. \tag{305}$$

4. Construct the midpoints $\{m_i\}_{i=1}^{n-1}$ by

$$m_i = \frac{z_{i+1}+z_i}{2}. \tag{306}$$

5. For $i = 2,3,\ldots,n-1$, compute

$$p_i = \Phi\left(\frac{m_i-\mu}{\sigma_z}\right) - \Phi\left(\frac{m_{i-1}-\mu}{\sigma_z}\right), \tag{307}$$

   where $\Phi(\cdot)$ is the CDF of the standard normal distribution.

6. For the endpoints,

$$p_1 = \Phi\left(\frac{m_1-\mu}{\sigma_z}\right), \tag{308}$$

$$p_n = 1 - \Phi\left(\frac{m_{n-1}-\mu}{\sigma_z}\right). \tag{309}$$

Remarks:

- The probability assigned to z_i is the probability that a draw from the normal distribution falls into the interval constructed around z_i.
- A larger n improves the approximation, though computational constraints may limit n.
- The choice of λ is critical, particularly for small n, so that certain properties of the original distribution are replicated.

43.2 Approximating an AR(1) Process via Tauchen's Method

Consider the AR(1) process:

$$z' = (1-\rho)\mu + \rho z + \varepsilon', \quad \varepsilon' \sim \text{i.i.d. } N(0, \sigma_\varepsilon^2). \tag{310}$$

The goal is to approximate this process with discrete states $\{z_i\}_{i=1}^n$ and transition probabilities $\{p_{ij}\}_{i,j=1}^n$.

Algorithm (Tauchen, 1986):

1. Set n, the number of realizations.
2. Recognize that the stationary distribution of z is $N(\mu, \sigma_z^2)$, where
$$\sigma_z = \frac{\sigma_\varepsilon}{\sqrt{1-\rho^2}}.$$
3. Define the bounds:
$$\bar{z} = \mu + \lambda\sigma_z, \tag{311}$$
$$\underline{z} = \mu - \lambda\sigma_z. \tag{312}$$
4. Set $\{z_i\}_{i=1}^n$ equally spaced:
$$z_i = \underline{z} + \frac{2\lambda\sigma_z}{n-1}(i-1), \quad i = 1, 2, \ldots, n. \tag{313}$$
5. Construct the midpoints $\{m_i\}_{i=1}^{n-1}$:
$$m_i = \frac{z_{i+1} + z_i}{2}. \tag{314}$$
6. Define the intervals $\{Z_i\}_{i=1}^n$:
$$Z_1 = (-\infty, m_1], \tag{315}$$
$$Z_i = [m_{i-1}, m_i], \quad i = 2, 3, \ldots, n-1, \tag{316}$$
$$Z_n = [m_{n-1}, \infty). \tag{317}$$
7. For $j = 2, 3, \ldots, n-1$, the transition probabilities are
$$p_{ij} = \Phi\left(\frac{m_j - (1-\rho)\mu - \rho z_i}{\sigma_\varepsilon}\right) - \Phi\left(\frac{m_{j-1} - (1-\rho)\mu - \rho z_i}{\sigma_\varepsilon}\right), \tag{318}$$
with
$$p_{i1} = \Phi\left(\frac{m_1 - (1-\rho)\mu - \rho z_i}{\sigma_\varepsilon}\right), \tag{319}$$
$$p_{in} = 1 - \Phi\left(\frac{m_{n-1} - (1-\rho)\mu - \rho z_i}{\sigma_\varepsilon}\right). \tag{320}$$

Remarks:

- There is no single answer for choosing n and λ. Monte Carlo experiments in Tauchen (1986) suggest that $n = 9$ and $\lambda = 3$ perform well even under high persistence.
- After selecting n based on computational feasibility, λ should be chosen such that key statistics of the approximated process match those of the original process.

43.3 Solving the Dynamic Programming Problem: Value Function Iteration

Assume that asset holdings are constrained to lie on a grid:

$$a \in \{a_1, a_2, \ldots, a_n\}, \tag{321}$$

and that the shock ε_t takes values in the finite set

$$E = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m\},$$

with a transition probability matrix Π. Given all parameter values, we set up an algorithm to compute the value function numerically.

Recursive Formulation: The value function is given by

$$v_1(a_i, \varepsilon_j) = \max_{1\le l\le n} \left\{ u\Big((1+r)a_i + w\varepsilon_j - a_l\Big) + \beta \sum_{r=1}^{m} \pi_{jr}\, v_0(a_l, \varepsilon_r) \right\}, \tag{322}$$

for $i = 1, \ldots, n$. Initially, $v_0(a_i, \varepsilon_j)$ is set to a matrix (often the zero matrix). Then, one compares v_0 and v_1 using a norm, for example,

$$d = \max_{i\in\{1,\ldots,n\}} |v_i^1 - v_i^0|. \tag{323}$$

If $d \le \varepsilon$, convergence is achieved; otherwise, update $v_0 = v_1$ and iterate further. Once convergence is reached, both the optimal value function and the policy function are obtained.

44 Heterogeneity in Macroeconomics: Economies with Idiosyncratic Risk and Incomplete Markets

44.1 Introduction

We now embed the consumption-savings problem into a general equilibrium framework following the Bewley/Huggett/Aiyagari approach. In this economy, a continuum of agents faces idiosyncratic earnings shocks. The individual consumer's problem provides a micro-foundation for the analysis of aggregate variables and the effects of micro-level policies. In a stationary equilibrium, aggregate variables are time-invariant.

A key question is how the interest rate r is determined in such an environment. The aggregate distribution of wealth matters (and not just the average), and this distributional aspect is crucial in a recursive competitive equilibrium framework. Following Aiyagari (1994), the model includes a production sector (a representative firm) that demands consumer savings and labor as inputs. Equilibrium in the asset market then yields an interest rate r such that $\beta(1+r) < 1$, reflecting precautionary motives.

44.2 Model Setup

44.2.1 Demographics and Preferences

The economy is populated by a continuum (measure one) of infinitely lived, ex-ante identical agents. Preferences are time-separable and defined over consumption streams:

$$U(c_0, c_1, \ldots) = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{324}$$

where u satisfies $u' > 0$, $u'' < 0$, and $\beta \in (0,1)$. The expectation is taken over future shock sequences. Agents supply labor inelastically.

44.2.2 Endowment

Each individual receives a stochastic endowment of efficiency units of labor $\varepsilon_t \in E \equiv \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N\}$, where the shocks follow a Markov process with transition probabilities

$$\pi(\varepsilon', \varepsilon) = \Pr(\varepsilon_{t+1} = \varepsilon' \mid \varepsilon_t = \varepsilon).$$

Shocks are i.i.d. across individuals. A law of large numbers applies, so $\pi(\varepsilon', \varepsilon)$ also represents the fraction of agents experiencing a given transition. With a unique invariant distribution $\Pi^*(\varepsilon)$, the aggregate endowment is constant:

$$H_t = \sum_{i=1}^{N} \varepsilon_i \, \Pi^*(\varepsilon_i), \quad \forall \, t. \tag{325}$$

44.2.3 Budget Constraint

For an individual i at time t, the budget constraint is:

$$c_t + a_{t+1} = (1 + r_t)a_t + w_t \varepsilon_t. \tag{326}$$

Wealth is held in the form of a one-period risk-free bond priced at one, with a return of $(1 + r_t)$ that is non-contingent on the realization of ε_{t+1}.

44.2.4 Liquidity Constraint

At every period t, agents face a borrowing limit:

$$a_{t+1} \geq -b, \tag{327}$$

where b is exogenous. Alternatively, one could assume that agents face the "natural" borrowing constraint, defined as the present value of the lowest possible realization of future earnings.

44.2.5 Technology

The representative competitive firm produces with CRS production function

$$Y_t = F(K_t, H_t)$$

with decreasing marginal returns in both inputs and standard Inada conditions. Physical capital depreciates at rate $\delta \in (0, 1)$.

44.2.6 Market Structure

Final good market (consumption and investment goods), labor market, and capital market are all competitive.

The aggregate resource constraint is given by

$$F(K_t, H_t) = C_t + I_t = C_t + K_{t+1} - (1 - \delta)K_t, \tag{328}$$

where capital letters represent aggregate variables.

45 Equilibrium

45.1 Recursive Competitive Equilibrium

The stationary equilibrium of this economy requires the distribution of agents across states to be invariant. However, individuals move up and down in the earnings and wealth distribution, so "social mobility" can be meaningfully defined. Recall that with complete markets (under certain conditions), there is no social mobility: initial rankings persist forever.

45.2 Mathematical Preliminaries

In what follows we characterize the individual and aggregate state as well as the evolution of the distribution over states.

45.2.1 Individual and Aggregate States

The individual is characterized by the pair (a, ε) — the individual states. The aggregate state of the economy is the distribution of agents across states, i.e. $\lambda(a, \varepsilon)$. Using the distribution, we can aggregate over individual decisions and compute aggregate variables and prices.

45.2.2 Probability Measure and State Space

We would like this object to be a probability measure, so we need to define an appropriate mathematical structure. The probability measure will permanently reproduce itself. It is in this sense that the economy is in a rest-point, i.e. a steady state.

Let $\bar{a}$ be the maximum asset holding in the economy, and for now assume that such an upper bound exists. Define the compact set

$$A \equiv [-b, \bar{a}]$$

of possible asset holdings and the countable set E as above.

Define the Cartesian product as the state space,

$$S \equiv A \times E,$$

with Borel σ-algebra $\mathcal{B}$ and typical subset $S \equiv A \times E$. The space $(S, \mathcal{B})$ is a measurable space, and for any set $S \in \mathcal{B}$, $\lambda(S)$ is the measure of agents in the set S. Finally, let Λ denote the set of all probability measures over $(S, \mathcal{B})$.

45.2.3 Transition Function and the Operator T^*

To characterize how individuals transit across states over time, we introduce a transition function. Define $Q((a, \varepsilon), A \times E)$ as the probability that an individual with current state (a, ε) transits to the set $A \times E$ next period. Formally,

$$Q : S \times \mathcal{B} \to [0, 1],$$

and with I denoting the indicator function and $a'(a, \varepsilon)$ the policy function (i.e. the optimal saving policy),

$$Q((a, \varepsilon), A \times E) = \sum_{\varepsilon' \in E} I_{a'(a,\varepsilon) \in A} \pi(\varepsilon', \varepsilon). \tag{329}$$

We then define the operator

$$T^* : \Lambda(S, \mathcal{B}) \to \Lambda(S, \mathcal{B}),$$

which maps distributions into distributions according to:

$$(T^* \lambda)(a, E) = \int_{A \times E} Q((a, \varepsilon), A \times E) d\lambda(a, \varepsilon). \tag{330}$$

This can also be written recursively as:

$$\lambda_{n+1}(a, E) = \int_{A \times E} Q((a, \varepsilon), A \times E) d\lambda_n(a, \varepsilon). \tag{331}$$

45.3 Recursive Formulation of the Household Problem

The household's problem is given by the recursive formulation:

$$v(a, \varepsilon; \lambda) = \max_{c,a'} \left\{ u(c) + \beta \sum_{\varepsilon' \in E} \pi(\varepsilon', \varepsilon) v(a', \varepsilon'; \lambda) \right\}, \tag{332}$$

subject to

$$c + a' = (1 + r(\lambda))a + w(\lambda)\varepsilon, \tag{333}$$

$$a' \geq -b. \tag{334}$$

Note that in the individual's dynamic program, λ is also a state variable as it is needed to compute market clearing prices. However, since we impose stationarity on the aggregate allocation, the distribution λ is time invariant. It does not change over time, and therefore it is not necessary to track its evolution explicitly as a state variable.

45.4 Definition of Stationary Recursive Competitive Equilibrium (RCE)

A stationary recursive competitive equilibrium consists of:

- A value function $v : S \to \mathbb{R}$,
- Household policy functions $a' : S \to \mathbb{R}$ and $c : S \to \mathbb{R}_+$,
- Firm policies for labor H and capital K,
- Prices r and w,
- A stationary measure $\lambda^* \in \Lambda$,

such that:

1. Given prices r and w, the policy functions a' and c solve the household's problem and v is the associated value function.
2. Given prices r and w, the firm chooses K and H optimally, satisfying

$$r + \delta = F_K(K, H), \tag{335}$$

$$w = F_H(K, H). \tag{336}$$

3. The labor market clears:

$$H = \int_{A \times E} \varepsilon \, d\lambda^*(a, \varepsilon). \tag{337}$$

4. The asset market clears:

$$K = \int_{A \times E} a'(a, \varepsilon) \, d\lambda^*(a, \varepsilon). \tag{338}$$

5. The goods market clears (redundant by Walras' Law):

$$F(K, H) = \delta K + \int_{A \times E} c(a, \varepsilon) \, d\lambda^*(a, \varepsilon). \tag{339}$$

6. For all $(A \times E) \in \mathcal{B}$, the invariant probability measure satisfies

$$\lambda^*(A \times E) = \int_{A \times E} Q((a, \varepsilon), A \times E) \, d\lambda^*(a, \varepsilon), \tag{340}$$

meaning that λ^* is a fixed point of the operator T^* such that $\lambda^* = T^*(\lambda)$.

45.5 Existence and Uniqueness

The strategy for establishing existence and uniqueness is as follows. An excess demand function is constructed for the asset market. One shows that this function is continuous and that it takes on both positive and negative values. With these conditions, there exists a price r such that the excess demand equals zero. The focus on the asset market is justified because the labor market clearing holds by construction (e.g., due to exogenous labor supply and the aggregate production technology) and because if asset markets clear, then goods markets clear by Walras' Law.

45.5.1 Compactness and the Feller Property

It can be shown that there exists an upper bound on the asset space, $\bar{a}$, when $\beta(1+r) < 1$ and utility is of the DARA class. Given that income shocks follow a discrete Markov process, the state space is compact. A transition function Q has the Feller Property if the operator T^* maps bounded and continuous functions into the space of bounded continuous functions, which is useful in proving existence.

45.5.2 Monotonicity and the Monotone Mixing Condition (MMC)

A transition function Q is monotone if for any bounded, continuous and increasing function $f : S \to \mathbb{R}$ the operator T^* produces a bounded, continuous and increasing function $T^*(f)$. The Monotone Mixing Condition (MMC) for uniqueness requires that an agent starting from any state $(a, \varepsilon) \in S$ will reach any arbitrary state $(a', \varepsilon') \in S$ within a finite time. This condition, sometimes referred to as the "American Dream, American Nightmare" condition, implies full mobility across states over time. For example, if an agent hypothetically starts from $(\bar{a}, \bar{\varepsilon})$ and draws $\underline{\varepsilon}$ for T periods (with T large), then by virtue of mean reversion the agent will eventually reach a neighborhood of $\underline{a}$. Similar reasoning applies for transitions from low to high states.

46 Calibration of the Model

To solve the model numerically, one first selects values for the parameters. Suppose the model's period is one year.

46.1 Technology Parameters

With a Cobb-Douglas production function, choose the capital share α equal to 1/3. Set the depreciation rate δ to 6%.

46.2 Preferences

Typically, CRRA utility is used. Let γ be the coefficient of relative risk aversion with acceptable values ranging between 1 and 5. Values at the lower end (e.g. $\gamma = 1$ or $\gamma = 2$) are most common.

46.3 Discount Rate

The discount rate β should be chosen so that the aggregate wealth-income ratio approximates that of the U.S. economy (around 3). In complete markets, the following condition holds:

$$\alpha K^{\alpha-1} H^{1-\alpha} - \delta = \left(\frac{1}{\beta} - 1\right) \Rightarrow \alpha \left(\frac{Y}{K}\right) - \delta = \frac{1}{\beta} - 1. \tag{341}$$

Thus,

$$\beta = \frac{1}{1 + \alpha \left(\frac{Y}{K}\right) - \delta} = 0.951. \tag{342}$$

In incomplete markets, the same β gives a slightly larger capital-output ratio due to extra precautionary saving, so one should set β slightly smaller.

46.4 Labor Income Process

To calibrate labor endowment shocks to replicate typical U.S. earnings dynamics, one may use data from the Panel Study of Income Dynamics (PSID). A reasonable approximation is an AR(1) process:

$$\ln y_t = \rho \ln y_{t-1} + \nu_t \quad \text{with } \nu_t \sim N(0, \sigma_\nu), \tag{343}$$

with $\rho = 0.95$ and $\sigma_\nu = 0.2$. More sophisticated estimates might include a transitory component, less persistent shocks, and a fixed individual component (e.g. education, ability).

46.5 Borrowing Constraint

If the natural borrowing constraint is inappropriate, one might calibrate it to match the observed fraction of agents with negative wealth (around 15% in the U.S. economy). This strategy, however, requires internal calibration.

47 Computing the Equilibrium

In this section we describe an algorithm based on finding a fixed point over the interest rate.

47.1 Algorithm Overview

Step 1: Initial Guess for Interest Rate

Fix an initial guess for the interest rate $r^0 \in (-\delta, \frac{1}{\beta} - 1)$. The superscript denotes the iteration number.

Step 2: Determination of the Wage Rate

Given the interest rate r^0, obtain the wage rate $w(r^0)$ using the CRS property of the production function (note that labor supply H is exogenously given with inelastic labor supply).

Step 3: Solving the Household Problem

Given prices $(r^0, w(r^0))$, solve the dynamic programming problem of the agent to obtain the policy functions $a'(a, \varepsilon; r^0)$ and $c(a, \varepsilon; r^0)$.

Step 4: Constructing the Invariant Distribution

Using the policy function $a'(a, \varepsilon; r^0)$ and the Markov transition over productivity shocks $\pi(\varepsilon', \varepsilon)$, construct the transition function $Q(r^0)$. By successive iterations over the transition, obtain the fixed point distribution $\lambda(r^0)$ conditional on the candidate interest rate r^0. In practice, this step is implemented by simulating a large number of households (e.g. 10,000) and tracking their states over time. At every period, compute cross-sectional moments (e.g. mean, variance, various percentiles) of the asset distribution. Stop when the moments converge, since for any given r, a unique invariant distribution is reached.

Step 5: Aggregate Capital Demand

Compute the aggregate demand for capital $K(r^0)$ from the firm's optimization:

$$K(r^0) = F_K^{-1}(r^0, \delta). \tag{344}$$

Step 6: Aggregate Asset Supply

Compute the aggregate asset supply:

$$A(r^0) = \int_{A\times E} a'(a,\varepsilon;r^0)d\lambda(a,\varepsilon). \tag{345}$$

This can be done using the data generated in Step 4.

Step 7: Market Clearing and Updating the Interest Rate

Compare $K(r^0)$ with $A(r^0)$ to verify the asset market clearing condition. If $A(r^0) > (<)K(r^0)$, then the next guess for the interest rate should be lower (higher), i.e. $r^1 < (>)r^0$. One update rule is:

$$r^1 = \frac{1}{2}\left\{r^0 + \left[F_K(A(r^0),H) - \delta\right]\right\}. \tag{346}$$

Note that r^0 and $F_K\big(A(r^0),H\big) - \delta$ are on opposite sides of the steady-state interest rate r^*.

Step 8: Iteration Until Convergence

Update the guess to r^1 and return to Step 1. Iterate until the interest rate converges, i.e.

$$|r^{n+1} - r^n| < \varepsilon, \tag{347}$$

for a small ε.

Step 9: Computing Equilibrium Statistics

Once convergence is achieved, compute all equilibrium statistics of interest (e.g., aggregate savings, inequality measures) from the simulated data obtained in Step 4.

47.2 Remarks on Numerical Implementation

47.2.1 Asset Grid Construction

Accuracy may be improved by placing more grid points near the lower bound $\underline{a}$, where the policy function is most nonlinear and exhibits the steepest slope. Near the upper bound $\bar{a}$, the policy function is nearly linear. With the borrowing constraint, saving policy functions may have kinks near the constraint. Maliar, Maliar, and Valli (2010) propose the following formula for choosing grid points:

$$a_i = \underline{a} + \left(\frac{i-1}{n_a-1}\right)^{\theta}(\bar{a} - \underline{a}) \quad \forall i = 1,\dots,n_a. \tag{348}$$

The parameter θ determines the concentration of grid points; $\theta = 1$ gives equally spaced nodes, $\theta > 1$ concentrates points near $\underline{a}$, and $\theta < 1$ concentrates them near $\bar{a}$.

47.2.2 Linear Interpolation

Modern methods often replace costly root-finding procedures with inexpensive interpolation methods. Suppose we have a grid G_x with associated function values $f(x_i)$ for each $x_i \in G_x$, and we wish to approximate $f(\hat{x})$ for some $\hat{x} \notin G_x$. Linear interpolation finds the interval $[x_i, x_{i+1}]$ such that $x_i \le \hat{x} \le x_{i+1}$ and computes:

$$f(\hat{x}) = \left(1 - \frac{\hat{x} - x_i}{x_{i+1} - x_i}\right) f(x_i) + \frac{\hat{x} - x_i}{x_{i+1} - x_i} f(x_{i+1}). \tag{349}$$

For convenience, denote this interpolation as

$$f(\hat{x}) = I(x \in G_x, f(x) \mid \hat{x}).$$

48 Heterogeneity in Macroeconomics: Incomplete Markets with Aggregate Uncertainty (Krusell and Smith, 1998)

48.1 Introduction

One can interpret a stationary equilibrium as the long-run outcome of an economy with individual-level heterogeneity and uncertainty but no aggregate uncertainty. However, for many macroeconomic questions, aggregate uncertainty is central—for example, business cycle analysis typically requires an aggregate TFP shock. With aggregate shocks, aggregate outcomes (e.g., aggregate savings and the joint distribution of wealth and income) become time-varying.

48.2 Model Setup

48.2.1 Demographics and Preferences

The economy is populated by a continuum of infinitely lived, ex-ante identical agents (of measure one) and a representative firm. Preferences are time separable and defined over consumption streams:

$$U(c_0, c_1, \ldots) = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{350}$$

with

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \quad \sigma > 1.$$

Each individual supplies labor inelastically.

48.2.2 Production

A representative firm operates a constant returns to scale production technology with capital and labor as inputs. The technology takes a Cobb-Douglas form:

$$zK^{\alpha}L^{1-\alpha},$$

where $\alpha \in (0, 1)$ and z is a productivity shock.

48.2.3 Uncertainty

There are two types of shocks:

- Idiosyncratic labor productivity shocks $\varepsilon \in \{\underline{\varepsilon}, \bar{\varepsilon}\}$ with $\underline{\varepsilon} < \bar{\varepsilon}$. In particular, in the low state $\underline{\varepsilon} = 0$ and in the high state $\bar{\varepsilon} = 1$.
- Aggregate productivity shocks $z \in \{\underline{z}, \bar{z}\}$ with $\underline{z} < \bar{z}$. The low state corresponds to a bust (recession) and the high state to a boom (recovery).

The stochastic state of the economy is therefore (ε, z). The joint process is modeled as a four-state Markov chain in which the distribution of idiosyncratic shocks depends on the aggregate shock, and the aggregate shock follows an autoregressive process. The conditional probability that z' is realized next period given current z is denoted $\pi_z(z' \mid z)$.

The joint conditional probability that an agent receives an individual shock ε' and aggregate shock z' next period given current shocks (ε, z) is denoted

$$\pi(z', \varepsilon' \mid \varepsilon, z).$$

The mass of agents with shock ε when the aggregate state is z is given by $\pi_\varepsilon(\varepsilon \mid z)$. It is assumed that

$$\sum_{z'}\sum_{\varepsilon'} \pi(\varepsilon', z' \mid \varepsilon, z) = 1, \tag{351}$$

$$\sum_{\varepsilon'} \pi(\varepsilon', z' \mid \varepsilon, z) = \pi_z(z' \mid z), \tag{352}$$

so that absent individual risk, the Markov chain reduces to the aggregate process.

Further assumptions include:

- Employment shocks are more likely when productivity is high:

$$\pi_\varepsilon(\underline{\varepsilon} \mid \bar{z}) < \pi_\varepsilon(\underline{\varepsilon} \mid \underline{z}).$$

- If an agent is unemployed in a recession, the probability of obtaining an employment shock $\bar{\varepsilon}$ is higher if the economy transitions to a recovery:

$$\pi(\bar{\varepsilon}, \underline{z} \mid \underline{\varepsilon}, \underline{z}) < \pi(\bar{\varepsilon}, \bar{z} \mid \underline{\varepsilon}, \underline{z}).$$

- If an agent is employed during a boom, the probability of remaining employed is higher if the economy stays in a boom than if it transitions to a bust:

$$\pi(\bar{\varepsilon}, \underline{z} \mid \bar{\varepsilon}, \bar{z}) < \pi(\bar{\varepsilon}, \bar{z} \mid \bar{\varepsilon}, \bar{z}).$$

48.2.4 State Variables

The individual state is (a, ε). There are also two aggregate states: the productivity shock z and the distribution λ of wealth and labor productivity shocks. These aggregate states determine current and future prices (wages and rental rates). Aggregate labor is given as a function of z since labor is supplied inelastically (with $\underline{\varepsilon} = 0$).

The firm optimally chooses capital K and labor L, setting marginal products equal to factor prices:

$$w(K, z) = (1-\alpha)z\left(\frac{K}{L(z)}\right)^{\alpha}, \tag{353}$$

$$r(K, z) = \alpha z\left(\frac{K}{L(z)}\right)^{\alpha-1} - \delta, \tag{354}$$

with $\delta < 1$ being the depreciation rate.

Aggregate capital is determined by the market clearing condition that aggregate savings equals the demand for capital:

$$K(\lambda) = \sum_{\varepsilon}\int_A a\,\lambda(a, \varepsilon)\,da. \tag{355}$$

Thus, aggregate capital depends on the distribution λ.

To forecast future prices, the agent needs to forecast next period's capital stock. Since this depends on the entire distribution λ, the law of motion for the distribution is needed.

Denote the law of motion of the distribution as

$$\lambda' = H(\lambda, z, z').$$

Given a policy function $g(a, \varepsilon)$, we can define the transition:

$$\lambda'(A \times E) = \int_{A\times E} Q((a, \varepsilon), A \times E \mid z, z')\,d\lambda, \tag{356}$$

where

$$Q((a,\varepsilon), A \times E \mid z, z') = \sum_{\varepsilon' \in E} \pi(\varepsilon', z' \mid \varepsilon, z)\, 1_{[g(a,\varepsilon) \in A]}. \tag{357}$$

Given this law, the agent computes next period's capital as:

$$K(\lambda') = \sum_{\varepsilon'} \int_{\mathbb{A}} a' \, \lambda'(a', \varepsilon')\, da'.$$

48.3 The Agents' Problem and Equilibrium

Agent's Problem

Model Setup:

$$v(a, \varepsilon; \lambda, z) = \max_{c,a'} \left\{ u(c) + \beta \sum_{\varepsilon'} \sum_{z'} \pi(\varepsilon', z' \mid \varepsilon, z) v(a', \varepsilon'; \lambda', z') \right\},$$

subject to

$$\begin{aligned} c + a' &= (1 + r(K(\lambda), z))a + w(K(\lambda), z)\varepsilon, \\ a' &\geq \underline{a}, \\ \lambda' &= H(\lambda, z, z'). \end{aligned}$$

Definition of Equilibrium

A recursive competitive equilibrium in this setting consists of:

- A value function v,
- Household savings policy function g and consumption policy function c,
- Firm policies for K and L,
- A law of motion for the distribution H,
- Prices w and r,

such that:

1. Given (w, r, G), the pair (c, g) solves the household's problem with value function v.
2. Given (w, r), the firm optimally chooses (K, L) by equating marginal products to factor prices.
3. The asset and labor markets clear (and by Walras' Law, so does the goods market).
4. The law of motion H is generated by the policy function g and the Markov transition function π as described above.

48.4 Computing the Equilibrium Under Aggregate Uncertainty

48.4.1 The Challenge of the Distribution

A key complication is that the value function v depends on λ, an infinite-dimensional object. Since prices depend on aggregate capital, which in turn depends on the full distribution λ, the problem of computing the law of motion $\lambda' = H(\lambda, z, z')$ is challenging.

48.4.2 Moment-Based Approximation

Any distribution can be represented by its moments. Let $\bar{m}$ be an M-dimensional vector containing the first M moments (e.g. mean, variance, skewness, kurtosis) of the wealth distribution. The law of motion can then be approximated by a function of these moments:

$$\bar{m}' = H_M(\bar{m}, z, z').$$

Krusell and Smith (1998) show that aggregate capital K (i.e. the first moment) is sufficient to forecast future prices with high accuracy. Therefore, one can specify a law of motion only for the first moment:

$$\ln K' = b_z^0 + b_z^1 \ln K,$$

where the coefficients b_z^0 and b_z^1 depend on the aggregate productivity shock.

The household problem then becomes:

$$v(a, \varepsilon; K, z) = \max_{c,a'} \left\{ u(c) + \beta \sum_{\varepsilon'} \sum_{z'} \pi(\varepsilon', z' \mid \varepsilon, z) v(a', \varepsilon'; K', z') \right\},$$

subject to

$$\begin{aligned} c + a' &= (1 + r(K, z))a + w(K, z)\varepsilon, \\ a' &\geq \underline{a}, \\ \ln K' &= b_z^0 + b_z^1 \ln K. \end{aligned}$$

48.4.3 Numerical Algorithm

The following steps outline the algorithm:

1. **Guess the coefficients:** Choose initial guesses for b_z^0 and b_z^1.

2. **Solve the household problem:** Obtain the decision rules $a'(a, \varepsilon; z, K)$ and $c(a, \varepsilon; z, K)$.

3. **Simulation:** Simulate the economy for N individuals over T periods (e.g. $N = 10,000$ and $T = 2000$). First, generate a sequence of aggregate shocks. Then, for each individual, simulate idiosyncratic shocks using the Markov chain $\pi(\varepsilon', z' \mid \varepsilon, z)$ and update asset holdings accordingly.

4. **Regression:** Discard the first T^0 periods (e.g. $T^0 = 500$) to eliminate dependence on initial conditions. Using the remaining data, run the regression

$$\ln K_{t+1} = \beta_z^0 + \beta_z^1 \ln K_t,$$

   and estimate (β_z^0, β_z^1).

5. **Update coefficients:** If $(b_z^0, b_z^1) \neq (\beta_z^0, \beta_z^1)$, update the guess and return to Step 1. When the guesses match the estimated coefficients for each $z \in \{z_g, z_b\}$, the agents' forecasting rule is consistent with equilibrium aggregation.

6. **Model Validation:** Compute measures of fit (e.g. R^2) for the regression. If adding additional moments (e.g. the second moment $m^2 = E(a_i^2)$) significantly improves the fit, include them and repeat the estimation until the fit is satisfactory.

48.4.4 Near-Aggregation Result

Krusell and Smith (1998) find that a law of motion based solely on the mean of the wealth distribution yields an excellent approximation. For instance, they obtain:

$$\ln K_{t+1} = 0.095 + 0.962 \ln K_t \quad \text{for } z = z_g,$$
$$\ln K_{t+1} = 0.085 + 0.965 \ln K_t \quad \text{for } z = z_b,$$

with an $R^2 = 0.999998$. This means that agents following this forecasting rule make only very small errors (e.g. the maximal error in forecasting the interest rate 25 years ahead is around 0.1%). This phenomenon is known as near-aggregation since the aggregate variables behave almost as in a complete-markets economy.

49 Introduction to Search Models

49.1 Unemployment Measurement and the Need for New Models

Unemployment is measured as the number of persons actively seeking work. Clearly, there is no counterpart to this concept in standard representative agent neoclassical growth models. In order to understand the behavior of the labor market, explain why unemployment fluctuates and how it is correlated with other key macroeconomic aggregates, and evaluate the efficacy of policies affecting the labor market, a new set of models is necessary. These models must incorporate heterogeneity to study equilibria where agents engage in different activities, such as job search, employment, and possibly leisure (when not in the labor force).

49.2 Labor Market Frictions and Early Approaches to Search

There must exist frictions which imply that it takes time for an agent to transit between unemployment and employment. Search models naturally incorporate such characteristics. Some early approaches to search and unemployment are presented in McCall (1970) and Phelps et al. (1970). These models are examples of "one-sided" search, analyzed in a partial equilibrium context where unemployed workers face a fixed distribution of wage offers. Later developments by Mortensen and Pissarides introduced two-sided search models (for a summary, see Pissarides 1990), in which workers and firms match in general equilibrium and wages are determined endogenously.

49.3 Empirical Facts of the Labor Market

49.3.1 Hires and Separations

Empirical evidence on hires and separations can be viewed at:

https://fred.stlouisfed.org/series/JTSHIL#0

49.3.2 The Procyclicality of Net Job Creation

The cyclicality in net job creation is documented in:

https://fred.stlouisfed.org/series/JTS1000HIL#0

49.3.3 Quits and Layoffs

For further insights into quits and layoffs, refer to:

https://fred.stlouisfed.org/series/JTS1000QUR#0

49.3.4 Job Finding Rate

Information on the job finding rate is available at:

https://fred.stlouisfed.org/series/LNS17100000#0

49.3.5 The Beveridge Curve

The Beveridge Curve, which illustrates the relationship between unemployment and job vacancies, is also a key empirical fact in the study of the labor market.

50 Two-Period McCall One-Sided Search Model

50.1 Model Motivation and Key Questions

This model examines decisions such as whether to accept a wage offer today or to continue searching for a better one, when it is optimal to quit a job, and whether some unemployed workers search more intensively than others.

50.2 Basic Setup

An unemployed worker receives a wage offer, w. The worker faces the choice of either accepting the offer and working in periods t and $t+1$, or rejecting the offer, collecting an unemployment benefit b, and waiting for another wage offer in period $t+1$, which is drawn from a distribution $F(w)$. The model assumes linear utility.

50.3 Objective Function and Reservation Wage

The worker's objective function is given by

$$U = \max\left\{w + w\beta,\ b + \beta\int_{\underline{w}}^{\bar{w}} \max\{w, b\}\, dF(w)\right\}.$$

The reservation wage, w_r, satisfies

$$(1+\beta)w_r = b + \beta\int_{\underline{w}}^{\bar{w}} \max\{w, b\}\, dF(w).$$

Assuming $\underline{w} > b$, this simplifies to

$$w_r = \frac{b + \beta\mu}{1+\beta},$$

where

$$\mu = \int_{\underline{w}}^{\bar{w}} w\, dF(w).$$

50.4 Comparative Statics

The comparative statics of the reservation wage are as follows:

$$\frac{\partial w_r}{\partial \beta} = \frac{\mu - b}{(1+\beta)^2} > 0,$$

$$\frac{\partial w_r}{\partial \mu} = \frac{\beta}{(1+\beta)} > 0,$$

$$\frac{\partial w_r}{\partial b} = \frac{1}{(1+\beta)} > 0.$$

50.5 Evolution of Unemployment

The evolution of unemployment is governed by job separations and job finding. Let s denote the separation rate and f the job-finding rate. Then, the change in unemployment is given by

$$u_{t+1} - u_t = -f u_t + s(1 - u_t).$$

In steady state, where $u_{t+1} = u_t = u^\star$, we have

$$u^\star = \frac{s}{s + f}.$$

51 Infinite Horizon and One-Sided Search Models

51.1 Infinite Horizon One-Sided Search Model

Consider a continuum of agents of unit mass, each with preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t c_t,$$

with $0 < \beta < 1$ (and with $\beta = \frac{1}{1+r}$, where r is the interest rate). It is assumed that there is no disutility from labor effort either on the job or in the process of searching for a job. The economy contains many different jobs which differ according to the wage, w, received by the worker.

51.2 A One-Sided Search Model: Setup and Value Functions

From the perspective of an unemployed agent, the distribution of wage offers is given by a probability distribution function $F(w)$ with associated density $f(w)$, where the support is $w \in [0, \bar{w}]$. For an employed agent receiving a wage w (and assuming that each job requires the input of one unit of labor per period), consumption equals w, as there are no saving opportunities. At the end of each period, an employed worker becomes unemployed with probability δ, the separation rate.

An unemployed worker receives an unemployment benefit b at the beginning of the period, followed by a wage offer which may be accepted or declined. It is assumed that $b < \bar{w}$ so that some job offers provide compensation higher than the unemployment insurance benefit. Let V_u denote the value of being unemployed and $V_e(w)$ denote the value of being employed at wage w, both measured at the end of the period.

51.3 Bellman Equations and Simplifications

The value functions are determined by the following Bellman equations:

$$V_u = \beta \left\{ b + \int_0^{\bar{w}} \max\left[V_e(w), V_u\right] f(w)\, dw \right\},$$

$$V_e(w) = \beta \left[w + \delta V_u + (1 - \delta) V_e(w) \right].$$

Notice that an employed agent remains employed (i.e., $V_e(w) \geq V_u$) if no separation occurs.

A useful simplification is obtained by dividing the first equation by β, substituting $\beta = \frac{1}{1+r}$, and subtracting V_u from both sides:

$$rV_u = b + \int_0^{\bar{w}} \max\left[V_e(w) - V_u, 0\right] f(w)\, dw \quad (3).$$

Similarly, the employed agent's value function becomes

$$rV_e(w) = w + \delta\left[V_u - V_e(w)\right] \quad (4).$$

51.4 Determination of the Reservation Wage

From equation (4), we can write

$$V_e(w) = \frac{w + \delta V_u}{r + \delta} \quad (5).$$

Since $V_e(w)$ is strictly increasing in w, there exists a reservation wage w^* such that $V_e(w) \geq V_u$ for $w \geq w^*$ and $V_e(w) \leq V_u$ for $w \leq w^*$. By definition, at w^* the worker is indifferent, so

$$V_e(w^*) = V_u,$$

which, using (5), implies

$$V_u = \frac{w^*}{r} \quad (6).$$

Substituting (5) and (6) into (3) yields

$$w^* = b + \int_0^{\bar{w}} \max\left[\frac{w - w^*}{r + \delta}, 0\right] f(w)\, dw,$$

or equivalently,

$$w^* = b + \frac{1}{r + \delta} \int_{w^*}^{\bar{w}} (w - w^*) f(w)\, dw.$$

This expression can be rewritten as

$$w^* = b + \frac{1}{r + \delta} \left\{ \int_{w^*}^{\bar{w}} w f(w)\, dw - w^*[1 - F(w^*)] \right\}.$$

After integrating by parts, we obtain

$$w^* = b + \frac{1}{r + \delta} \left\{ \bar{w} - w^* F(w^*) - \int_{w^*}^{\bar{w}} F(w)\, dw - w^*[1 - F(w^*)] \right\},$$

which simplifies to

$$w^* = b + \frac{1}{r + \delta} \int_{w^*}^{\bar{w}} [1 - F(w)]\, dw \quad (7).$$

51.5 Interpretation and Comparative Statics

Equation (7) determines the reservation wage w^*, and by construction, $w^* > b$. Although it is intuitive that an unemployed worker would not accept a job offering a wage below the unemployment insurance benefit, the worker also rejects wage offers that exceed b by only a small margin. This is because the worker may prefer to remain unemployed and continue collecting b in the hope of obtaining a significantly higher wage offer in the future.

An increase in the unemployment insurance benefit, b, has the effect of increasing w^* (i.e., $\frac{dw^*}{db} > 0$). This is because a higher b lowers the cost of being unemployed, making the worker more selective. Changes in the interest rate r and the separation rate δ affect the reservation wage as follows:

$$\frac{dw^*}{dr} < 0.$$

An increase in r implies that future payoffs are discounted at a higher rate, reducing the worker's willingness to wait for a better wage offer. Similarly, if the separation rate δ increases, the expected lifetime of a job decreases, reducing the attractiveness of holding out for a higher wage. In this case, even though the gap between $V_e(w)$ and V_u narrows as indicated by

$$V_e(w) - V_u = \frac{w - w^*}{r + \delta},$$

the net effect is that unemployed workers become less selective and lower their reservation wage.

51.6 Employment and Unemployment Dynamics

Let u_t denote the fraction of agents who are unemployed in period t. The flow into employment is given by the fraction of unemployed agents times the probability of transitioning from unemployment to employment, namely $u_t[1 - F(w^*)]$. Conversely, the flow from employment to unemployment is determined by job separations, given by $(1 - u_t)\delta$. Hence, the law of motion for unemployment is

$$u_{t+1} = u_t - u_t[1 - F(w^*)] + (1 - u_t)\delta = u_t[F(w^*) - \delta] + \delta \quad (8).$$

Since $|F(w^*) - \delta| < 1$, the sequence $\{u_t\}$ converges to a steady state u, which is obtained by setting $u_{t+1} = u_t = u$ in (8) and solving:

$$u = \frac{\delta}{\delta + 1 - F(w^*)} \quad (9).$$

It follows that an increase in the separation rate δ raises the unemployment rate, and a higher reservation wage (which reduces the job-finding rate, $1 - F(w^*)$) similarly increases the unemployment rate. Note that while an increase in δ directly increases unemployment, it also has the indirect effect of reducing w^*, which tends to lower the unemployment rate; thus, the net effect of a change in δ is ambiguous.

51.7 General Equilibrium Considerations

The partial equilibrium approach described above does not account for the fact that if job vacancies are posted by firms, then the wage offer distribution becomes endogenous. In equilibrium, this distribution is influenced by the rate at which unemployed workers accept various jobs and by the composition of jobs posted by firms. Moreover, the financing of unemployment insurance benefits—typically through lump-sum taxes on employed agents—introduces additional general equilibrium effects that are not captured in the partial equilibrium framework.

51.8 Conclusion

These notes provide an overview of both two-period and infinite-horizon one-sided search models, highlighting the key elements such as the reservation wage, comparative statics, and the evolution of unemployment. The models underscore the importance of incorporating heterogeneity and market frictions to better understand labor market dynamics. The general equilibrium extensions of these models further suggest the interconnectedness of wage distributions, job vacancy postings, and policy instruments like unemployment insurance.

52 Search and Unemployment: A Two-Sided Search Model: Mortensen-Pissarides

52.1 Introduction

In many macroeconomic analyses, general equilibrium search models of unemployment are employed to determine wages endogenously and to assess the impact of policy. These models capture the dual role of search in both the labor market and firm behavior, providing a rich framework for analysis.

52.2 The Model

52.2.1 Basic Setup

We consider a continuum of agents with unit mass, each having preferences given by

$$E_0 \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t c_t,$$

with $r > 0$. There is also a continuum of firms with an objective function given by

$$E_0 \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t (\pi_t - x_t),$$

where π_t denotes the firm's profits (which are consumed by the firm) and x_t represents the disutility from posting a vacancy during period t. Goods are perishable and agents save nothing from period to period.

52.2.2 Labor Market and Matching

Let u_t denote the mass of workers who are unemployed in period t, with $1 - u_t$ representing those matched with firms and thus employed. Similarly, let v_t denote the mass of firms posting vacancies. In each period, matches between unemployed workers and vacancies occur according to the matching function

$$m_t = m(u_t, v_t),$$

where $m(\cdot, \cdot)$ is continuous, increasing in both arguments, concave, homogeneous of degree 1, and satisfies $m(0, v) = m(u, 0) = 0$ for all $u, v \geq 0$.

By homogeneity, the probability that an individual unemployed worker finds a job is given by

$$\frac{m(u_t, v_t)}{u_t} = m\left(1, \frac{v_t}{u_t}\right),$$

and the probability that an individual firm fills a vacancy is

$$\frac{m(u_t, v_t)}{v_t} = m\left(\frac{u_t}{v_t}, 1\right).$$

For convenience, we define the labor market tightness as

$$\theta_t \equiv \frac{v_t}{u_t}.$$

An increase in θ_t raises the job-finding probability for workers while reducing the probability that a firm fills a vacancy. We also assume that

$$\lim_{\theta \to 0} m(1/\theta, 1) = \lim_{\theta \to \infty} m(1, \theta) = 1.$$

Each firm possesses a technology that produces y units of output with one unit of labor per period, with any other quantity yielding zero output.

52.2.3 Employment, Separation, and Payoffs

Each worker has one unit of time available each period. Upon matching with a firm and agreeing on a contract, the worker and the firm jointly produce y units of output until separation occurs. Separation happens each period with probability δ. While unemployed, a worker receives unemployment insurance compensation b each period (the financing of b is not modeled). A firm posting a vacancy incurs a cost κ per period, while a firm that is not posting a vacancy and is unmatched receives zero utility.

52.2.4 Bargaining

We now confine attention to a steady state equilibrium where $u_t = u$ and $v_t = v$ for all t. When a match occurs, a wage w is negotiated. Let $W(w)$ denote the value to the worker of being employed at wage w and $J(y-w)$ the value to the firm of the match. Let U denote the value to a worker of remaining unemployed and V the value to a firm of posting a vacancy. An agreement occurs only if

$$W(w) - U \geq 0 \quad \text{and} \quad J(y-w) - V \geq 0,$$

with these differences representing the surpluses from the match for the worker and the firm, respectively. The total surplus is given by

$$S = W(w) + J(y-w) - U - V.$$

A tractable approach is to assume that the worker and firm engage in Nash bargaining:

$$w = \arg\max_{w'} [W(w') - U]^{\alpha} [J(y-w') - V]^{1-\alpha},$$

subject to the constraints

$$W(w') - U \geq 0, \quad J(y-w') - V \geq 0,$$

where $0 \leq \alpha \leq 1$ measures the worker's bargaining power. Although this optimization is not solved by any individual agent, its solution characterizes the bargaining outcome. Ignoring the constraints momentarily, the first-order condition simplifies to

$$\alpha\, W' [J(y-w) - V] - (1-\alpha)\, J'(y-w) [W(w) - U] = 0 \quad (1).$$

For the worker, the value of being employed at wage w is given by

$$W(w) = \frac{1}{1+r} \Big[w + (1-\delta) W(w) + \delta U \Big] \quad (2),$$

and for the firm,

$$J(y-w) = \frac{1}{1+r} \Big[y - w + (1-\delta) J(y-w) + \delta V \Big] \quad (3).$$

After rearrangement, these become

$$rW(w) = w + \delta[U - W(w)] \quad (4),$$

and

$$rJ(y-w) = y - w + \delta[V - J(y-w)] \quad (5).$$

Thus, we obtain

$$W(w) = \frac{w + \delta U}{r + \delta} \quad (6), \qquad J(y-w) = \frac{y - w + \delta V}{r + \delta} \quad (7),$$

and consequently,

$$W'(w) = J'(y-w) = \frac{1}{r+\delta} \quad (8).$$

Substituting into (1) leads to

$$\alpha \Big[J(y-w) - V \Big] - (1-\alpha) \Big[W(w) - U \Big] = 0 \quad (9).$$

52.2.5 Equilibrium Conditions

The values for an unemployed worker and for a firm posting a vacancy are given by the following Bellman equations. Since all jobs pay the same wage in equilibrium, we denote by W the equilibrium value of employment and by J the value of a match. Then,

$$U = \frac{1}{1+r}\Big\{b + m(1,\theta)W + \Big[1 - m(1,\theta)\Big]U\Big\},$$

and

$$V = \frac{1}{1+r}\Big\{-\kappa + m(1/\theta,1)J + \Big[1 - m(1/\theta,1)\Big]V\Big\}.$$

After simplification, these yield

$$rU = b + m(1,\theta)(W-U) \quad (10),$$

and

$$rV = -\kappa + m(1/\theta,1)(J-V) \quad (11).$$

Under the free entry condition, firms earn zero on posting vacancies:

$$V = 0 \quad (12).$$

Defining the total surplus from a match as

$$S = W + J - U - V = W + J - U \quad (13),$$

equation (9) implies

$$W - U = \alpha S \quad (14),$$

so that Nash bargaining ensures the worker receives a fraction α of the total surplus. Hence, the firm's surplus is given by

$$J - V = (1-\alpha)S \quad (15).$$

By combining (4) and (5) with (10) and (11) and then subtracting, we obtain

$$r(W + J - U - V) = y - b + \delta(U - W - J) - m(1,\theta)(W-U) \quad (16).$$

Using (13), (14), and (15) to simplify (16) yields

$$S = \frac{y-b}{r+\delta+m(1,\theta)\alpha} \quad (17),$$

and, from (11) and (15) with $V = 0$,

$$S = \frac{\kappa}{(1-\alpha)m(1/\theta,1)} \quad (18).$$

Equations (17) and (18) jointly determine S and θ.

Let $F(\theta)$ be the right-hand side of (17) and $G(\theta)$ the right-hand side of (18). These functions are continuous with $F'(\theta) < 0$ and $G'(\theta) > 0$ and satisfy

$$F(0) = \frac{y-b}{r+\delta}, \quad G(0) = \frac{\kappa}{1-\alpha}, \quad F(\infty) = \frac{y-b}{r+\delta+\alpha}, \quad G(\infty) = \infty.$$

An equilibrium exists if and only if

$$\kappa < \frac{(1-\alpha)(y-b)}{r+\delta},$$

that is, if the cost of posting a vacancy is sufficiently small. When this condition holds, the equilibrium is unique with $S = S^*$ and $\theta = \theta^*$ satisfying

$$\frac{y-b}{r+\delta+\alpha} < S^* < \frac{y-b}{r+\delta}.$$

Since $S > 0$, both the worker and the firm earn positive surplus in equilibrium, confirming that every meeting between a worker and a firm results in a successful match.

52.3 Wage Determination and Value Functions

From (5) and (15), given S the wage is determined by

$$w = y - (r + \delta)(1 - \alpha)S \quad (19).$$

Given S and w, equation (4) yields

$$W = \frac{w + \delta\alpha S}{r} \quad (20),$$

and using $W - U = \alpha S$ in conjunction with (20) results in

$$U = \frac{w + (\delta - r)\alpha S}{r} \quad (21).$$

52.3.1 Steady State Dynamics

In the steady state, the flow of workers transitioning from unemployment to employment is $u\,m(1, \theta)$, while the flow from employment to unemployment is $(1 - u)\delta$. Equating these flows gives

$$u\,m(1, \theta) = (1 - u)\delta.$$

Thus, the steady state unemployment rate is

$$u = \frac{\delta}{m(1, \theta) + \delta} \quad (22),$$

and, given $\theta = \frac{v}{u}$, the mass of vacancies is

$$v = u\theta = \frac{\delta\theta}{m(1, \theta) + \delta} \quad (23).$$

52.4 Comparative Statics and Experiments

52.4.1 Effect of an Increase in Productivity

An increase in y (aggregate productivity) raises the total surplus S and the labor market tightness θ. As a consequence, unemployment falls due to an increased job-finding rate. Differentiating (23) with respect to θ yields

$$\frac{dv}{d\theta} = \frac{m(1, \theta) + \delta - m_2(1, \theta)\theta}{[m(1, \theta) + \delta]^2} = \frac{m_1(1, \theta) + \delta}{[m(1, \theta) + \delta]^2} > 0,$$

where the homogeneity-of-degree-one property of the matching function is used. It can also be shown that the wage w increases. Overall, an increase in y makes posting vacancies more attractive, raising v and θ, and lowering unemployment. This mechanism reflects the negative correlation between u and v along the Beveridge curve.

52.4.2 Effect of an Increase in Unemployment Insurance

An increase in unemployment insurance compensation b has the opposite effect. A higher b reduces the total surplus S, making vacancy posting less attractive for firms; hence, both v and θ fall. The job-finding rate decreases, causing the unemployment rate u to rise. Moreover, equation (19) implies that the wage w increases because the attractiveness of unemployment forces firms to offer higher wages.

52.4.3 Effect of an Increase in the Separation Rate

An increase in the separation rate δ reduces both S and θ. Unemployment u rises directly from the higher separation rate and indirectly due to the lower job-finding rate associated with a lower θ. Although the decline in θ tends to reduce vacancies v, the direct effect of a higher δ may increase v, so that both u and v can rise simultaneously. In contrast to productivity shocks, shocks to δ may produce a positive correlation between u and v, a feature not observed in the data, suggesting that productivity shocks better capture the qualitative aspects of the labor market.